3) A communication system transmits binary information using the antipodal pulses $g(t)$ or $-g(t)$, where $g(t)$ is shown below

a) Find the impulse response of the matched filter.

b) Sketch the output of the matched filter when the pulse $g(t)$ is applied to the input. Show all important details.

c) Suppose that two-sided noise power spectral density is $10^{-12}$ watts/Hz. What is the bit energy-to-noise ratio?

d) What is the probability of bit error. You can leave your answer in terms of a $Q$-function.

\[ h(t) = g(T-t) \]

\[ \frac{E_b}{N_0} = \frac{A^2 T}{2} = \frac{16 \times 10^{-6} \times 10^{-6}}{2 \times 10^{-12}} = 8 = 9.03 \text{ dB} \]

d) $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(4)$
1) Random Processes: Suppose that $X(t)$ and $Y(t)$ are wide-sense stationary random processes with means $\mu_X$ and $\mu_Y$, autocorrelation functions $\phi_{XX}(\tau)$ and $\phi_{YY}(\tau)$, power spectral densities $\Phi_{XX}(f)$ and $\Phi_{YY}(f)$, cross-correlation function $\phi_{XY}(\tau)$ and cross power spectral density $\Phi_{XY}(f)$.

Consider the sum process

$$Z(t) = X(t) + Y(t)$$

a) 4 marks: Derive an expression for the autocorrelation function of $Z(t)$ in terms of $\phi_{XX}(\tau)$, $\phi_{YY}(\tau)$ and $\phi_{XY}(\tau)$.

b) 3 marks: Suppose that $X(t)$ and $Y(t)$ are uncorrelated meaning that their crosscovariance function $\mu_{XY}(\tau) = \phi_{XY}(\tau) - \mu_X \mu_Y = 0$. What is the autocorrelation function of $Z(t)$?

c) 2 marks: Suppose further that $X(t)$ and $Y(t)$ are uncorrelated and $\mu_X = \mu_Y = 0$. What is the autocorrelation function of $Z(t)$?

d) 1 marks: What is the power spectral density of $Z(t)$ in part c)?

\[
\varphi_{ZZ}(\tau) = \mathbb{E} \left[ Z(t)Z(t+\tau) \right] = \mathbb{E} \left[ (X(t) + Y(t))(X(t+\tau) + Y(t+\tau)) \right] = \mathbb{E} \left[ X(t)X(t+\tau) + Y(t)X(t+\tau) + \right.
\]

\[
+ X(t)Y(t+\tau) + Y(t)Y(t+\tau) \right] = \mathbb{E} \left[ X(t)X(t+\tau) \right] + \mathbb{E} \left[ Y(t)Y(t+\tau) \right] + \mathbb{E} \left[ X(t)Y(t+\tau) \right] + \mathbb{E} \left[ Y(t)X(t+\tau) \right]
\]

use

\[
\varphi_{xx}(\tau) = \varphi_{yx}(\tau) = \varphi_{yx}(-\tau) = \varphi_{yy}(\tau) = \varphi_{yx}(-\tau) = \varphi_{xx}(\tau) + \varphi_{yx}(\tau)
\]

b) $\varphi_{ZZ}(\tau) = \varphi_{xx}(\tau) + \varphi_{yy}(\tau) + 2\mu_x \mu_y$

c) $\phi_{zz}(\tau) = \phi_{xx}(\tau) + \phi_{yy}(\tau)$

d) $\Phi_{zz}(f) = \Phi_{xx}(f) + \Phi_{yy}(f)$
1) Consider a wide sense stationary random process \( X(t) \) having mean \( \mu_X \) and autocorrelation function \( \phi_{XX}(\tau) \) and power spectrum \( S_{XX}(f) \). The random process \( X(t) \) is used to construct another random process \( Z(t) \) as follows:

\[
Z(t) = X(t) - X(t - 2T)
\]

where all quantities are real valued.

a) 1 mark What is the output mean \( \mu_Z \)?

b) 3 marks What is the autocorrelation of \( Z(t) \), \( \phi_{ZZ}(\tau) \), in terms of \( \phi_{XX}(\tau) \)?

c) 2 marks What is the output power spectrum \( S_{ZZ}(f) \)?

d) 3 marks What is the cross-correlation of \( X(t) \) and \( Z(t) \), \( \phi_{XZ}(\tau) \), in terms of \( \phi_{XX}(\tau) \)?

d) 1 mark If the input process \( X(t) \) was a Gaussian random process, is \( Z(t) \) a Gaussian random process?

\[
\begin{align*}
\text{a) } \mu_Z & = E[Z(t)] = E[X(t) - X(t-2T)] \\
& = E[X(t)] - E[X(t-2T)] = \mu_X - \mu_X = 0
\end{align*}
\]

\[
\begin{align*}
\text{b) } \phi_{ZZ}(\tau) & = E[Z(t)Z(t+\tau)] \\
& = E[(X(t) - X(t-2T))(X(t+\tau) - X(t-2T+\tau))] \\
& = E[X(t)X(t+\tau)] - E[X(t-2T)X(t+\tau)] \\
& \quad - E[X(t)X(t-2T+\tau)] + E[X(t-2T)X(t-2T+\tau)] \\
& = \phi_{XX}(\tau) - \phi_{XX}(\tau+2T) - \phi_{XX}(\tau-2T) + \phi_{XX}(\tau) \\
& = 2 \phi_{XX}(\tau) - \phi_{XX}(\tau+2T) - \phi_{XX}(\tau-2T)
\end{align*}
\]

\[
\begin{align*}
\text{c) } S_{ZZ}(f) & = 2S_{XX}(f) - S_{XX}(f)e^{-j4\pi f T} - S_{XX}(f)e^{j4\pi f T} \\
& = 2S_{XX}(f)\left(1 - \frac{e^{j4\pi f T} + e^{-j4\pi f T}}{2}\right) \\
& = 2(1 - \cos 4\pi f T)S_{XX}(f) \\
& = 4 \sin^2(2\pi f T)S_{XX}(f)
\end{align*}
\]
d) \[ \phi_{x_2} (t) = E \left[ x(t) z(t + 2) \right] \]
\[ = E \left[ x(t) (x(t+2) - x(t+1-2T)) \right] \]
\[ = E \left[ x(t)x(t+2) \right] - E \left[ x(t)x(t+1-2T) \right] \]
\[ = \phi_{xx} (2) - \phi_{xx} (1-2T) \]

e) Yes, at any time \( t \),
\[ z(t,1) = x(t,1) + x(t,1-2T) \]

The sum of Gaussian random variables is a Gaussian random variable.
1) **Random Processes**: Suppose that the inputs $X(t)$ and $Y(t)$ to a multiplier are independent random processes with power spectral densities

\[
S_X(f) = 5 \text{rect} \left( \frac{f}{10} \right)
\]
\[
S_Y(f) = 2 \text{rect} \left( \frac{f}{6} \right)
\]

a) 7 marks: Compute and sketch the power spectral density at the output of the multiplier.

b) 3 marks: What is the total power at the output?

\[a) \text{ We have seen from Homework that } \]
\[Z(t) = X(t)Y(t) \text{ has autocorrelation function } \]
\[
\phi_{z,z}(\tau) = \phi_{xx}(\tau) \phi_{yy}(\tau)
\]

Hence, $S_z(f) = S_x(f) \ast S_y(f)$, where $\ast$ denotes convolution.

You can use graphical convolution.

\[\begin{array}{c}
\text{Sketch of } S_x(f) \\
\text{5} \\
-5 \quad 0 \quad 5 \\
\end{array}
\]

\[\begin{array}{c}
\text{Sketch of } S_y(f) \\
\text{2} \\
-3 \quad 0 \quad 3 \\
\end{array}
\]

\[\text{Sketch of } S_z(f) \\
\text{60} \\
-10 \quad -5 \quad 0 \quad 1 \quad 2 \\
\end{array}
\]
b) total power

\[ P_T = \int_{-\infty}^{\infty} S_z(f) \, df \]

\[ = \frac{1}{2} (6)(60) + (4)(60) + \frac{1}{2} (6)(60) \]

\[ = 600 \text{ watts} \]
2) The random process $X(t)$ with mean $\mu_X$ and autocorrelation function $\phi_{XX}(\tau)$ is applied to the filters $g(t)$ and $h(t)$ as shown below. The output processes are $Y(t)$ and $Z(t)$.

a) Find the output means $\mu_Y$ and $\mu_Z$ in terms of $\mu_X$.

b) Find the cross-correlation function

$$\phi_{YZ}(\tau) = E[Y(t)Z(t+\tau)]$$

in terms of $\phi_{XX}(\tau)$, $g(t)$ and $h(t)$.

\[
\begin{align*}
\mu_Y &= G(0)\mu_X \\
\mu_Z &= H(0)\mu_X
\end{align*}
\]

\[
\phi_{YZ}(\tau) = E[Y(t)Z(t+\tau)]
= \mathbb{E} \left[ \int_{-\infty}^{\infty} g(\alpha) X(t-\alpha) d\alpha \int_{-\infty}^{\infty} h(\beta) X(t+\tau-\beta) d\beta \right]
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha) h(\beta) \mathbb{E} \left[ X(t-\alpha) X(t+\tau-\beta) \right] d\alpha d\beta
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha) h(\beta) \phi_{XX}(\tau-\beta+\alpha) d\alpha d\beta
= \int_{-\infty}^{\infty} h(\beta) \int_{-\infty}^{\infty} g(\alpha) \phi_{XX}(\tau+\alpha-\beta) d\alpha d\beta
= \left\{ \int_{-\infty}^{\infty} g(\alpha) \phi_{XX}(\tau+\alpha) d\alpha \right\} \ast h(\tau)
= g(-\tau) \ast \phi_{XX}(\tau) \ast h(\tau)
\]
3) A communication system transmits binary information using the antipodal pulses $g(t)$ or $-g(t)$, where $g(t)$ is shown below.

a) Find the impulse response of the matched filter.

b) Sketch the output of the matched filter when the pulse $g(t)$ is applied to the input. Show all important details.

c) Suppose that two-sided noise power spectral density is $10^{-12}$ watts/Hz. What is the bit energy-to-noise ratio?

d) What is the probability of bit error. You can leave your answer in terms of a $Q$-function.

\[
\begin{align*}
A & = 1 \times 10^{-6} \\
A & = 4 \times 10^{-3}
\end{align*}
\]

\[
A^2 T = 16 \times 10^{-6} \times 10^{-6} = 8 = 9.03 \text{ dB}
\]

\[
P_b = Q \left( \sqrt{\frac{2E_b}{N_0}} \right) = Q(4)
\]
2) Consider a binary communication system that transmits information using the pulse

\[ g(t) = A[-u(t) + 2u(t - T/2) - u(t - T)] \]

according to the mapping rule

\[
\begin{align*}
\text{"0"} & \quad \rightarrow \quad -g(t) \\
\text{"1"} & \quad \rightarrow \quad +g(t)
\end{align*}
\]

The "0"s and "1"s are transmitted with equal probability, and the channel is an AWGN channel, with a two-sided noise power spectral density of \( N_0/2 \) watts/Hz.

a) 3 marks Determine and sketch the filter \( h(t) \) that is matched to \( g(t) \).

b) 3 marks Determine and sketch the overall pulse \( p(t) = g(t) * h(t) \) for the filter you found in part (a), labelling all important points.

c) 2 marks Determine the pulse energy, \( E_p \), and the energy per bit, \( E_b \).

d) 2 marks What is the probability of bit error in terms of \( E_b/N_0 \)?
b) You can use graphical convolution

You can verify

\[ p(t) = \begin{cases} 
0, & t \leq 0 \\
-\frac{A^2 t}{2}, & 0 \leq t \leq T/2 \\
3A^2 t - 2A^2 T, & \frac{T}{2} \leq t \leq T \\
-3A^2 t + 4A^2 T, & T \leq t \leq \frac{3T}{2} \\
-2A^2 T + A^2 t, & \frac{3T}{2} \leq t \leq 2T \\
0, & t \geq 2T 
\end{cases} \]

\[ E = A^2 T, \quad E_b = \frac{1}{2} E + \frac{1}{2} E = E \]

\[ P_b = Q\left(\sqrt{\frac{2E_b}{I_0}}\right) \]
2) Consider the pair of pulses $g_1(t)$ and $g_2(t)$ shown in the figure below.

![Graphs of $g_1(t)$ and $g_2(t)$](image)

a) 4 marks Sketch the matched filters for these two pulses, $h_1(t)$ and $h_2(t)$.

b) 3 marks Sketch the waveforms at the output of the filters $h_1(t)$ and $h_2(t)$ if the input is $g_1(t)$.

c) 3 marks Sketch the waveforms at the output of the filters $h_1(t)$ and $h_2(t)$ if the input is $g_1(t) + g_2(t)$. 

![Graphs of $h_1(t)$ and $h_2(t)$](image)

![Graphs of $g_1(t) \ast h_1(t)$ and $g_2(t) \ast h_2(t)$](image)
2) Matched Filters: Consider the two pulses \( g_1(t) \) and \( g_2(t) \) shown below.

\[
g_1(t) \quad \text{and} \quad g_2(t)
\]

a) 4 marks Determine and sketch the corresponding matched filters \( h_1(t) \) and \( h_2(t) \).

b) 3 marks Sketch the outputs of the two matched filters when the pulse \( g_1(t) \) is applied at their input.

c) 3 marks Sketch the outputs of the two matched filters when the pulse \( g_2(t) \) is applied at their input.