Aids Allowed: Course text, calculator
Attempt all questions
Questions are of equal value

DATE: Wednesday October 8, 2012.
TIME: 2:05pm - 2:55pm
INSTRUCTOR: Prof. G.L. Stüber

Solution

\[ N = 31 \]
\[ \bar{X} = 18.5 \]
\[ \sigma = 6.1 \]
1) Matched Filters: Consider the two matched filters shown below

a) 4 marks: Find the pulses \(g_1(t)\) and \(g_2(t)\) to which these pulses are matched.

b) 2 marks: Find \(G_1(f)\) and \(G_2(f)\).

c) 1 mark: What is the energy in \(g_1(t)\) and \(g_2(t)\)?

d) 3 marks: Find the output of filter \(h_1(t) - h_2(t)\) to the pulse \(g_1(t)\).

\[
\begin{align*}
\text{a)} & \quad g_1(t) = A \quad \text{for} \quad 0 \leq t \leq T \\
\text{b)} & \quad G_1(f) = AT \sin \left( \frac{ft}{2} \right) e^{-j \pi ft} \\
\text{c)} & \quad G_2(f) = -\frac{AT}{2} \sin \left( \frac{ft}{2} \right) e^{-j \pi ft/2} + \frac{AT}{2} \sin \left( \frac{ft}{2} \right) e^{-j 3 \pi ft/2} \\
\end{align*}
\]

\[
\begin{align*}
G_2(f) & = -jAT \sin \left( \frac{ft}{2} \right) e^{-j 5 \pi ft/8} \left( e^{j \pi ft/8} - e^{-j \pi ft/8} \right) \\
& = AT \sin \left( \frac{ft}{2} \right) \sin \left( \frac{\pi ft}{8} \right) e^{-j (5 \pi ft/8 + \pi/2)}
\end{align*}
\]
c) \( E = A^2 T \)

d) \( h_1(t) - h_2(t) \)

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[Graph showing a square pulse and a triangle pulse]
2) Random Processes: A wide-sense stationary Gaussian random process $X(t)$ with autocorrelation function $\phi_{XX}(\tau)$ and power density spectrum $\Phi_{XX}(f)$ is input to the linear time-invariant system shown below. The filter $h(t)$ has impulse response

$$h(t) = e^{-\alpha t}u(t)$$

![System Diagram]

a) 3 marks Find the overall transfer function of the system, $H_{\text{eff}}(f)$, where $Y(f) = H_{\text{eff}}(f)X(f)$.

b) 3 marks Find the power spectral density of the output $\Phi_{YY}(f)$ in terms of $\Phi_{XX}(f)$.

c) 2 marks Which frequencies cannot be present in $\Phi_{YY}(f)$?

d) 2 marks Is the output $Y(t)$ a Gaussian random process? Is it wide-sense stationary?

\[
\begin{align*}
H_{\text{eff}}(f) &= H(f) e^{-j2\pi f T} + H(f) e^{j2\pi f T} \\
&= \frac{2}{X + j2\pi f} \left( e^{j2\pi f T} + e^{-j2\pi f T} \right) \\
&= \frac{2 \cos(2\pi f T)}{X + j2\pi f}
\end{align*}
\]
b) \[ \Phi_{yy}(f) = |H_{eff}(f)|^2 \Phi_{xx}(f) \]
\[ = \frac{A \cos^2(2\pi fT)}{x^2 + (2\pi f)^2} \Phi_{xx}(f) \]

c) \[ |H_{eff}(f)|^2 = 0 \text{ when } \cos(2\pi fT) = 0 \]
\[ 2\pi fT = \frac{\pi n}{2}, \quad n \text{ odd} \]
\[ f = \frac{n}{4T}, \quad n \text{ odd} \]

d) Yes, b/c input process is Gaussian
Yes, b/c input process is WSS
3) **Error Probability:** Suppose that multi-hop communication system is modeled as the serial concatenation of 3 identical binary symmetric channels (BSCs) each having crossover probability $p$ as shown in the figure below. Since the overall channel has two inputs and two outputs, the overall channel can be modeled as a single BSC with crossover probability $p_{\text{eff}}$ as shown in the figure.

![Diagram of BSCs](image)

a) **4 marks:** Find $p_{\text{eff}}$ in terms of $p$.

b) **2 mark:** Suppose each hop uses sends “0s” and “1s” with equal probability using the pulses $g(t)$ and $-g(t)$, respectively, where $g(t)$ has energy $E$. The waveform channel is an additive white Gaussian noise channel with two-sided noise spectral density $N_0/2$ watts/Hz. What is $p$ in terms of the per hop $E_b/N_0$?

c) **1 mark:** If $E_b/N_0 = 3$ dB find $p_{\text{eff}}$ for part b) above.

d) **3 marks:** Extend the result in part a) to a serial concatenation of $n$ BSCs each having crossover probability $p$.

\[
\begin{align*}
a) & \quad P_{\text{eff}} = 3p(1-p)^2 + p^3 & \{\text{one of 3 hops in error}\} \\
b) & \quad p = \Phi(\sqrt{\frac{2E_b}{N_0}}) \\
c) & \quad p = \Phi(\sqrt{4}) = \Phi(2) = 1 - 0.97725 = 0.02275 \\
d) & \quad P_{\text{eff}} = \sum_{\ell=1}^{n} \binom{n}{\ell} p^\ell (1-p)^{n-\ell} & \{\text{prob of odd number of hops in error}\}
\end{align*}
\]