1) Consider the random process

\[ X(t) = A \cos(2\pi f_c t + \Theta), \quad -\infty \leq t \leq \infty \]

where \( A \) and \( f_c \) are constants and \( \Theta \) has the probability density function

\[ f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, & 0 \leq \theta \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \]

4  a) Find the mean and variance of this random process.

4  b) Obtain its autocorrelation function.

\[ \frac{2A}{\pi} \left[ \cos(2\pi f_c t) - \sin(2\pi f_c t) \right] \]

2  c) Is the process wide sense stationary? Strictly stationary?

5  a) \[ \mu_X(t) = E \left[ A \cdot \cos(2\pi f_c t + \Theta) \right] = \int_{-\pi/2}^{\pi/2} A \cdot \cos(2\pi f_c t + \Theta) \frac{2}{\pi} \, d\Theta \]

\[ \mu_X(t) = \frac{2A}{\pi} \sin(2\pi f_c t + \Theta) \bigg|_{-\pi/2}^{\pi/2} = \frac{2A}{\pi} \left[ \sin(2\pi f_c t + \pi/2) - \sin(2\pi f_c t) \right] \]

\[ \sigma_X^2(t) = E[(X(t) - \mu_X(t))^2] = \Phi_{XX}(0) - \mu_X^2(t) \]

\[ \sigma_X^2(t) = \frac{A^2}{2} - \frac{A^2}{\pi} \sin(4\pi f_c t) - \frac{4A^2}{\pi^2} \left[ \cos(2\pi f_c t) - \sin(2\pi f_c t) \right]^2 \]

\[ = \frac{A^2}{2} - \frac{A^2}{\pi} \sin(4\pi f_c t) - \frac{4A^2}{\pi^2} \left[ 1 - 2\cos(2\pi f_c t) \cdot \sin(2\pi f_c t) \right] \]

\[ = \frac{A^2}{2} - \frac{4A^2}{\pi^2} + \frac{4A^2}{\pi^2} \sin(4\pi f_c t) \]

6  b) \[ E[X(t)X(t+\tau)] = E[A \cdot \cos(2\pi f_c t + \Theta) \cdot A \cdot \cos(2\pi f_c t + \Theta)] \]

\[ = \frac{A^2}{2} E \left[ \cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta) \right] \]

\[ = \frac{A^2}{2} \cos(2\pi f_c \tau) + \frac{A^2}{2} \int_{-\pi/2}^{\pi/2} \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta) \frac{2}{\pi} \, d\Theta \]

\[ = \frac{A^2}{2} \cos(2\pi f_c \tau) + \frac{A^2}{\pi} \cdot \frac{2}{2} \cdot \sin(4\pi f_c t + 2\pi f_c \tau + 2\Theta) \bigg|_{-\pi/2}^{\pi/2} \]

\[ \Phi_{XX}(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau) - \frac{A^2}{\pi} \sin(4\pi f_c t + 2\pi f_c \tau) \]

C) The process is not W.S.S. because \( \mu_X(t) \) is a function of time; therefore, it is not strict sense stationary either.
2) The power spectral density of a signal is shown in the figure below.

3 a) Find the total power in the signal.

3 b) Find the amount of power contained in the frequency range 5 to 10 kHz.

4 c) Suppose the signal is applied to an ideal low pass filter with a bandwidth of 5 kHz. Find the power spectral density at the filter output and the total power in the output signal.

\[ S_x(f) \]

\[ 10^{-6} f^2 \text{ watts/Hz} \]

\[ -10 \text{ kHz} \]

\[ 10 \text{ kHz} \]

\[ f \]

\[ 8) \text{ Total Power} = \int_{-\infty}^{\infty} S_x(f) \, df = 2 \int_{0}^{10 \text{ kHz}} 10^{-6} f^2 \, df = \frac{2 \times 10^{-6}}{3} \left( \frac{10^6}{3} \right)^{\frac{3}{2}} = \frac{2 \times 10^{-6}}{3} \left( \frac{1}{3} \right)^{\frac{3}{2}} = 6.66 \times 10^{-5} \text{ watts} \]

\[ 6) \text{ Power in the 5 to 10 kHz range} = \int_{-5 \text{ kHz}}^{5 \text{ kHz}} S_x(f) \, df + \int_{10 \text{ kHz}}^{10 \text{ kHz}} S_x(f) \, df \\
= 2 \int_{5 \text{ kHz}}^{10 \text{ kHz}} 10^{-6} f^2 \, df = \frac{2}{3} \times 10^6 \left( \frac{10^6}{3} \right)^{\frac{3}{2}} = \frac{2}{3} \left[ 10^6 - (5^6) \right] = 5.833 \text{ watts} \]

\[ S_y(f) \]

\[ 10^{-6} f^2 \]

\[ -5 \text{ kHz} \]

\[ 5 \text{ kHz} \]

\[ \text{Total Power of } Y = \int_{-5 \text{ kHz}}^{5 \text{ kHz}} 10^{-6} f^2 \, df = \text{Result 6) - Result 8) } \]

\[ = 8.333 \times 10^{-6} \text{ watts} \]

3
3) A matched filter is described by the figure below

5 a) Find the impulse response of the matched filter.

5 b) Find the pulse shape to which this filter is matched.

\[ h_{mf}(t) = [\delta(t) - \delta(t-T)] \ast \mathcal{F}^{-1}\left\{ \frac{1}{j2\pi f} \right\} \]

\[ = [\delta(t) - \delta(t-T)] \ast \left\{ \frac{1}{2} \text{sgn}(t) \right\} \]

\[ = \frac{1}{2} \text{sgn}(t) - \frac{1}{2} \text{sgn}(t-T) \]

\[ = \begin{array}{c}
\text{rect} \left( \frac{t}{T} \right) \end{array} \]

6) pulse it is matched to is \( \text{rect} \left( \frac{T-t-T/2}{T} \right) = \text{rect} \left( \frac{T/2-t}{T} \right) \), same as \( \text{rect} \left( \frac{t-T/2}{T} \right) \)

\[ \text{inverted and shifted by T} \]