EE4601 Communication Systems

Week 11

Linear Zero Forcing Equalization

Equalization

- The cascade of the transmit filter $g(t)$, channel $c(t)$, receiver filter $h(t)$ yields the overall pulse
$$p(t) = g(t) * c(t) * h(t)$$
- The signal at the output of the matched filter is
$$y(t) = \sum_k a_k p(t - kT) + n(t)$$
and the sampled output is
$$\begin{aligned} y_n = y(nT) &= \sum_k a_k p_{n-k} + n_n \\ &= \sum_k p_k a_{n-k} + n_n \end{aligned}$$
- Assume a causal, finite-length, channel such that $p(t) = 0$ for $t < 0$ and $t > LT$.
- The discrete-time channel $p_n = p(nT)$, can be represented by the vector
$$\mathbf{p} = (p_0, p_1, \ldots, p_L)$$

Equalization

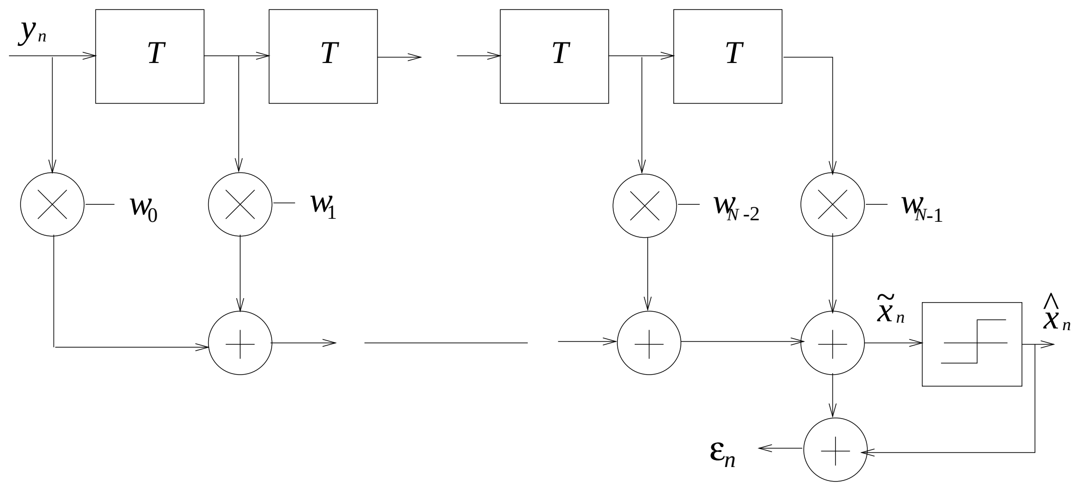
- An equalizer is a digital filter that is used to mitigate the effects of inter-symbol interference that is introduced by a time dispersive channel.
- The tap co-efficients of the equalizer are denoted by the vector

$$\mathbf{w} = (w_0,\ w_1,\ \cdots,\ w_{N-1})^T$$

  where N is the number of equalizer taps.
- If the equalizer is used to process the sampled outputs of the receiver matched filter, then the output of the equalizer is

$$\tilde{x}_n = \sum_{j=0}^{N-1} w_j y_{n-j}$$

Linear Transversal Equalizer



Overall Discrete-time Model

- The overall channel and equalizer can be represented by a overall digital filter with impulse response

$$\mathbf{q} = (q_0, q_1, \ \ldots, \ q_{N+L-1})^T$$

where

$$\begin{aligned} q_n &= \sum_{j=0}^{N-1} w_j p_{n-j} \\ &= \mathbf{w}^T \mathbf{p}(n) \end{aligned}$$

with

$$\mathbf{p}(n) = (p_n, p_{n-1}, p_{n-2}, \ldots, p_{n-N+1})^T$$

and $p_i = 0, i < 0, i > L$. That is, $\mathbf{q}$ is the discrete convolution of $\mathbf{p}$ and $\mathbf{w}$.

Perfect Equalization

- Let the component of $\mathbf{p}$ of greatest magnitude be denoted by p_{d_1}. Note that we may have $d_1 \neq 0$.
- Let the number of equalizer taps be equal to $N = 2d_2 + 1$ where d_2 is an integer.
- Perfect equalization means that

$$\mathbf{q} = \mathbf{e}_d = (\underbrace{0,\ 0,\ \ldots,\ 0}_{d \text{ zeroes}},\ 1,\ 0,\ \ldots,\ 0,\ 0)^T$$

  where d zeroes precede the "1" and d is an integer representing the overall delay, a parameter to be optimized.
- Unfortunately, perfect equalization is difficult to achieve and does not always yield the best performance.

Zero Forcing Equalizer

- With a zero-forcing (ZF) equalizer, the tap coefficients $\mathbf{w}$ are chosen to minimize the peak distortion of the equalized channel, defined as

$$D_{\mathrm{p}} = \frac{1}{|q_d|} \sum_{\substack{n=0 \\ n \neq d}}^{N+L-1} |q_n - \hat{q}_n|$$

  where $\hat{\mathbf{q}} = (\hat{q}_0,\ \ldots,\ \hat{q}_{N+L-1})^T$ is the *desired equalized channel* and the delay d is a positive integer chosen to have the value $d = d_1 + d_2$.

- Lucky showed that if the initial distortion without equalization is less than unity, i.e.,

$$D = \frac{1}{|p_{d_1}|} \sum_{\substack{n=0 \\ n \neq d_1}}^{L} |p_n| < 1 \ ,$$

  then D_p is minimized by those N tap values which simultaneously cause $q_j = \hat{q}_j$ for $d - d_2 \leq j \leq d + d_2$. However, if the initial distortion before equalization is greater than unity, the ZF criterion is not guaranteed to minimize the peak distortion.

Zero Forcing Equalizer

- For the case when $\hat{\mathbf{q}} = \mathbf{e}_d^T$ the equalized channel is given by

$$\mathbf{q} = (q_0, \ldots, q_{d_1-1}, 0, \ldots, 0, 1, 0, \ldots, 0, q_{d_1+N}, \ldots, q_{N+L-1})^T \ .$$

- In this case the equalizer forces zeroes into the equalized channel and, hence, the name "zero-forcing equalizer."

- If the ZF equalizer has an infinite number of taps it is possible to select the tap weights so that $D_{\mathrm{p}} = 0$, i.e., $\boldsymbol{q} = \hat{\boldsymbol{q}}$. Assuming that $\hat{q}_n = \delta_{n0}$ this condition means that

$$Q(z) = 1 = C(z)G(z) \ .$$

Therefore,

$$C(z) = \frac{1}{G(z)}$$

and the ideal ZF equalizer has a discrete transfer function that is simply the inverse of overall channel $G(z)$.

Equalizer Tap Solution

- For a known channel impulse response, the tap gains of the ZF equalizer can be found by the direct solution of a simple set of linear equations. To do so, we form the matrix

$$\mathbf{P} = [\mathbf{p}(d_1), \ldots, \mathbf{p}(d), \ldots, \mathbf{p}(N + d_1 - 1)]$$

  and the vector

$$\tilde{\mathbf{q}} = (\hat{q}_{d_1}, \ldots, \hat{q}_d, \ldots, \hat{q}_{N+d_1-1})^T \quad .$$

- Then the vector of optimal tap gains, $\mathbf{w}_{\text{op}}$, satisfies

$$\mathbf{w}_{\text{op}}^T \mathbf{P} = \tilde{\mathbf{q}}^T \longrightarrow \mathbf{w}_{\text{op}} = (\mathbf{P}^{-1})^T \tilde{\mathbf{q}} \ .$$

Example

- Suppose that a system has the channel vector

$$\mathbf{p} = (0.90, -0.15, 0.20, 0.10, -0.05)^T \ ,$$

where $p_i = 0, i < 0, i > 4$. The initial distortion before equalization is

$$D = \frac{1}{|p_0|}\sum_{n=1}^{4}|p_n| \ = \ 0.5555$$

and, therefore, the minimum distortion is achieved with the ZF solution.

- Suppose that we wish to design a 3-tap ZF equalizer. Since p_0 is the component of $\mathbf{p}$ having the largest magnitude, $d_1 = 0$ and the optimal equalizer delay is $d = 1$. The desired response is $\hat{\mathbf{q}} = \mathbf{e}_1^T$ so that $\tilde{\mathbf{q}} = (0,\ 1,\ 0)^T$.

Example

- We then construct the matrix

$$\begin{aligned} \mathbf{P} &= [\mathbf{p}(0), \mathbf{p}(1), \mathbf{p}(2)] \\ &= \begin{bmatrix} 0.90 & -0.15 & 0.20 \\ 0.00 & 0.90 & -0.15 \\ 0.00 & 0.00 & 0.90 \end{bmatrix} \end{aligned}$$

  and obtain the optimal tap solution

$$\mathbf{w}_{\rm op} = (\mathbf{P}^{-1})^T \tilde{\mathbf{q}} = (0,\ 1.11111,\ -0.185185)^T \ .$$

  The overall response of the channel and equalizer is

$$\mathbf{q} = (\ 0.0,\ 1.0,\ 0.0,\ 0.194,\ 0.148,\ -0.037,\ -0.009,\ 0, \ldots)^T \ .$$

- Hence, the distortion after equalization is

$$D_{\min} = \frac{1}{|q_0|} \sum_{n=1}^{6} |q_n - \hat{q}_n| = 0.388 \ .$$

Adaptive Solution

- In practice, the channel impulse response is unknown to the receiver and a known finite length sequence **a** is used to train the equalizer.
- During this training mode, the equalizer taps can be obtained by using the following steepest-descent recursive algorithm:

$$w_j^{n+1} = w_j^n + \alpha \epsilon_n a^*_{n-j-d_1} \ , \qquad j = 0, \ldots, N-1 \ , \tag{1}$$

where

$$\begin{aligned} \epsilon_n &= a_{n-d} - \tilde{a}_n \\ &= a_{n-d} - \sum_{i=0}^{N-1} w_i y_{n-i} \end{aligned} \tag{2}$$

is the error sequence, $\{w_j^n\}$ is the set of equalizer tap gains at epoch n.

- α is an adaptation step-size that can be optimized to trade off convergence rate and steady state bit error rate performance.

Adaptive Solution

- Fact: The adaptation rule in (1) attempts to force the crosscorrelations $\epsilon_n a^*_{n-j-d_1}$, $j = 0, \ldots, N-1$, to zero.
- To see that this leads to the desired solution we note

$$\begin{aligned}\frac{1}{2}\mathrm{E}[\epsilon_n a^*_{n-j-d_1}] &= \frac{1}{2}\mathrm{E}[a_{n-d}a^*_{n-j-d_1}] - \frac{1}{2}\sum_{i=0}^{N-1}\sum_{\ell=0}^{L} w_i c_\ell \mathrm{E}[a_{n-i-\ell}a^*_{n-j-d_1}] \\ &= \sigma_a^2\left(\delta_{d_2-j} - \sum_{i=0}^{N-1} w_i p_{j+d_1-i}\right) \\ &= \sigma_a^2(\delta_{d_2-j} - q_{j+d_1})\ , \quad j = 0,\ 1,\ \ldots,\ N-1\ , \qquad (3)\end{aligned}$$

where $\sigma_a^2 = \frac{1}{2}\mathrm{E}[|a_k|^2]$.

- Fact: The conditions $\frac{1}{2}\mathrm{E}[\epsilon_n a^*_{n-j-d_1}] = 0$ are satisfied when $q_d = 1$ and $q_i = 0$ for $d - d_2 \leq i < d$ and $d < i \leq d + d_2$, which is the zero forcing solution. Note the ensemble average over the noise and the data symbol alphabet.

Adaptive Solution

- After training the equalizer, a decision-feedback mechanism is typically employed where the sequence of symbol decisions $\hat{\mathbf{a}}$ is used to update the tap coefficients. This mode is called the data mode and allows the equalizer to track variations in the channel vector $\mathbf{c}$. In the data mode,

$$w_j^{n+1} = w_j^n + \alpha \epsilon_n \hat{a}^*_{n-j-d_1} \ , \qquad j = 0, \ldots, N-1 \ ,$$

where the error term ϵ_n in (2) becomes

$$\epsilon_n = \hat{a}_{n-d} - \sum_{i=0}^{N-1} c_i y_{n-i}$$

and, again, $\hat{a}_{n-d}$ is the decision on the equalizer output $\tilde{a}_n$ delayed by d samples.