EE4601
Communication Systems

Week 4

Ergodic Random Processes, Power Spectrum
Linear Systems

Ergodic Random Processes

An **ergodic** random process is one where time averages are equal to ensemble averages. Hence, for all $g(\mathbf{X})$ and $\mathbf{X}$

$$\begin{aligned}\mathrm{E}[g(\mathbf{X})] &= \int_{-\infty}^{\infty} g(\mathbf{X}) p_{\mathbf{X}(t)}(\mathbf{x}) d\mathbf{x} \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g[\mathbf{X}(t)] \mathrm{d}t \\ &= < g[\mathbf{X}(t)] > \end{aligned}$$

For a random process to be ergodic, it must be strictly stationary. However, not all strictly stationary random processes are ergodic.

A random process is **ergodic in the mean** if

$$< X(t) >= \mu_X$$

and **ergodic in the autocorrelation** if

$$< X(t)X(t+\tau) >= \phi_{XX}(\tau)$$

Example (cont'd)

Recall the random process

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_c are constants, and Θ is a uniformly distributed random phase.

$$p_\Theta(\theta) = \begin{cases} 1/(2\pi)\ , & 0 \le \theta \le 2\pi \\ 0\ , & \text{elsewhere} \end{cases}$$

The time average mean of $X(t)$ is

$$< X(t) >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(2\pi f_c t + \theta)dt = 0$$

In this example $\mu_X(t) =< X(t) >$, so the random process $X(t)$ is ergodic in the mean.

N.B. Make sure you understand the difference between the *time average* and *ensemble average*.

Example (cont'd)

The time average autocorrelation of $X(t)$ is

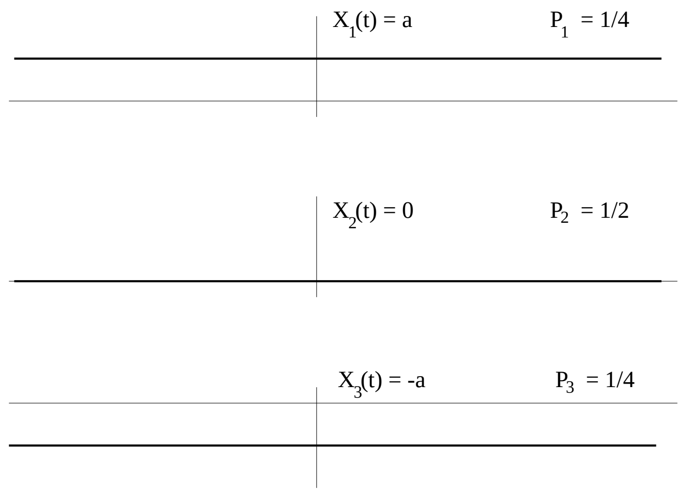
$$
\begin{aligned}
< X(t)X(t+\tau) > &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cos(2\pi f_c t + \theta) dt \\
&= \lim_{T\to\infty} \frac{A^2}{4T} \int_{-T}^{T} \left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\right] dt \\
&= \frac{A^2}{2} \cos(2\pi f_c \tau)
\end{aligned}
$$

The random process $X(t)$ is ergodic in the autocorrelation.

It follows that the random process $X(t)$ in this example is *ergodic in the mean and autocorrelation.*

Example

Consider the random process shown below.



Example (cont'd)

For this example, the *ensemble* and *time average* means are, respectively,

$$\begin{aligned} \mu_X &= \mathrm{E}[X(t)] = 0 \\ \langle X(t) \rangle &= \begin{cases} a & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -a & \text{with probability } 1/4 \end{cases} \end{aligned}$$

Hence, $X(t)$ is *not ergodic in the mean.*

The *ensemble* and *time average* autocorrelations are

$$\begin{aligned} \phi_{XX}(\tau) &= \mathrm{E}[X(t)X(t+\tau)] = a^2(1/4) + 0(1/2) + (-a)^2(1/4) = a^2/2 \\ \langle X(t)X(t+\tau) \rangle &= \begin{cases} a^2 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/2 \end{cases} \end{aligned}$$

Hence, $X(t)$ is *not ergodic in the autocorrelation.*

Example (cont'd)

Note that

$$\begin{aligned} \mathrm{E}[\langle X(t)\rangle] &= \mu_X \\ \mathrm{E}[\langle X(t)X(t+\tau)\rangle] &= \phi_{XX}(\tau) \end{aligned}$$

Because of this property $\langle X(t)\rangle$ and $\langle X(t)X(t+\tau)\rangle$ are said to provide *unbiased estimates* of μ_X and $\phi_{XX}(\tau)$, respectively.

Power Spectral Density

The power spectral density (psd) of a random process $X(t)$ is the Fourier transform of its autocorrelation function, i.e.,

$$\begin{aligned}\Phi_{XX}(f) &= = \int_{-\infty}^{\infty} \phi_{XX}(\tau) e^{-j2\pi f\tau} d\tau \\ \phi_{XX}(\tau) &= \int_{-\infty}^{\infty} \Phi_{XX}(f) e^{j2\pi f\tau} df \ \ .\end{aligned}$$

We have seen that $\phi_{XX}(\tau)$ is real and even. Therefore, $\Phi_{XX}(-f) = \Phi_{XX}(f)$ meaning that $\Phi_{XX}(f)$ is also real and even.

The total power (ac + dc), P, in a random process $X(t)$ is

$$P = \mathrm{E}[X^2(t)] = \phi_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f) df$$

a famous result known as **Parseval's theorem**.

Example

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_v are constants and

$$p_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}\ , & -\pi \leq \theta \leq \pi \\ 0\ , & \text{elsewhere} \end{cases}$$

We have seen before that

$$\phi_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Hence,

$$\begin{aligned} \Phi_{XX}(f) &= \frac{A^2}{2}\mathcal{F}[\cos(2\pi f_c \tau)] \\ &= \frac{A^2}{4}\left(\delta(f - f_c) + \delta(f + f_c)\right) \end{aligned}$$

Properties of $\Phi_{XX}(f)$

1. $\Phi_{XX}(0) = \int_{-\infty}^{\infty} \phi_{XX}(\tau) d\tau$
2. $\int_{0-}^{0+} \Phi_{XX}(f) df =$ dc power
3. $\phi_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f) df =$ total power
4. $\Phi_{XX}(f) \geq 0$ for all f. Power is never negative.
5. $\Phi_{XX}(f) = \Phi_{XX}(-f)$ (even function) if $X(t)$ is a real random process.
6. $\Phi_{XX}(f)$ is always real.

Discrete-time Random Processes

Consider a discrete-time real random process X_n, that consists of an ensemble of sample sequences $\{x_n\}$.
The ensemble mean of X_n is defined as

$$\mu_{X_n} = \mathrm{E}[X_n] = \int_{-\infty}^{\infty} x_n f_{X_n}(x_n) dx_n$$

The ensemble autocorrelation of X_n is

$$\phi_{XX}(n,k) = \mathrm{E}[X_n X_k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_n X_k f_{X_n,X_k}(x_n, x_k) dx_n dx_k$$

For a wide-sense stationary discrete-time real random process, we have

$$\begin{aligned} \mu_{X_n} &= \mu_X, \quad \forall n \\ \phi_{XX}(n,k) &= \phi_{XX}(n-k) \end{aligned}$$

From Parseval's theorem, the total power in the process X_n is

$$P = \mathrm{E}[X_n^2] = \phi_{XX}(0)$$

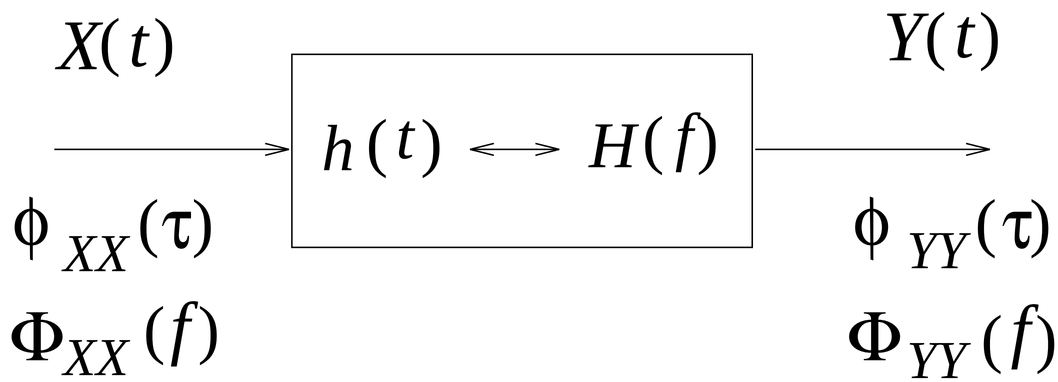
Power Spectrum of Discrete-time RP

The power spectrum of the real wide-sense stationary discrete-time random process X_n is the discrete-time Fourier transform of the autocorrelation function, i.e.,

$$\begin{aligned}\Phi_{XX}(f) &= \sum_{n=-\infty}^{\infty} \phi_{XX}(n) e^{-j2\pi fn} \\ \phi_{XX}(n) &= \int_{-1/2}^{1/2} \Phi_{XX}(f) e^{j2\pi fn} df\end{aligned}$$

Observe that the power spectrum $\Phi_{XX}(f)$ is periodic in f with a period of unity. In other words $\Phi_{XX}(f) = \Phi_{XX}(f+k)$, for $k = \pm 1, \pm 2, \ldots$ This is a characteristic of any discrete-time sequence. For example, one obtained by sampling a continuous-time random process $X_n = x(nT_s)$, where T_s is the sample period.

Linear Systems



Linear Systems

Suppose that the input to the linear system $h(t)$ is a wide sense stationary random process $X(t)$, with mean μ_X and autocorrelation $\phi_{XX}(\tau)$.

The input and output waveforms are related by the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau \ .$$

Hence,

$$Y(f) = H(f)X(f) \ .$$

The output mean is

$$\mu_Y = \int_{-\infty}^{\infty} h(\tau)\mathrm{E}[X(t-\tau)]d\tau = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau = \mu_X H(0) \ .$$

This is just the mean (dc component) of the input signal multiplied by the dc gain of the filter.

Linear Systems

The output autocorrelation is

$$\begin{aligned}
\phi_{YY}(\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\beta)X(t-\beta)d\beta \int_{-\infty}^{\infty} h(\alpha)X(t+\tau-\alpha)d\alpha\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\mathrm{E}\left[X(t-\beta)X(t+\tau-\alpha)\right] d\beta d\alpha \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\phi_{XX}(\tau-\alpha+\beta)d\beta d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta-\alpha)d\alpha d\beta \\
&= \left\{\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta)d\beta\right\} * h(\tau) \\
&= h(-\tau) * \phi_{XX}(\tau) * h(\tau) \ .
\end{aligned}$$

Taking transforms, the output psd is

$$\begin{aligned}
\Phi_{YY}(f) &= H^*(f)\Phi_{XX}(f)H(f) \\
&= |H(f)|^2\,\Phi_{XX}(f) \ .
\end{aligned}$$

Cross-correlation and Cross-covariance

If $X(t)$ and $Y(t)$ are each wide sense stationary and jointly wide sense stationary, then

$$\begin{aligned}\boldsymbol{\phi}_{XY}(t, t+\tau) &= \mathrm{E}[X(t)Y(t+\tau)] = \boldsymbol{\phi}_{XY}(\tau) \\ \boldsymbol{\mu}_{XY}(t, t+\tau) &= \boldsymbol{\mu}_{XY}(\tau) = \boldsymbol{\phi}_{XY}(\tau) - \mu_x \mu_y\end{aligned}$$

The crosscorrelation function $\phi_{XY}(\tau)$ has the following properties.

1. $\phi_{XY}(\tau) = \phi_{YX}(-\tau)$
2. $|\phi_{XY}(\tau)| \leq \frac{1}{2}[\phi_{XX}(0) + \phi_{YY}(0)]$
3. $|\phi_{XY}(\tau)|^2 \leq \phi_{XX}(0)\phi_{YY}(0)$ if $X(t)$ and $Y(t)$ have zero mean.

Example

Consider the linear system shown in the previous example. The crosscorrelation between the input process $X(t)$ and the output process $Y(t)$ is

$$\begin{aligned}
\phi_{YX}(\tau) &= \mathrm{E}[Y(t)X(t+\tau)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha X(t+\tau)\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)\mathrm{E}\left[X(t-\alpha)X(t+\tau)\right]d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\phi_{XX}(\tau+\alpha)d\alpha \\
&= h(-\tau) * \phi_{XX}(\tau)
\end{aligned}$$

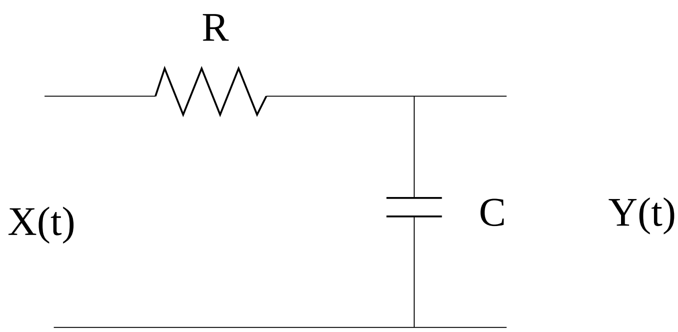
The cross power spectral density is

$$\Phi_{YX}(f) = H^*(f)\Phi_{XX}(f)$$

Note also that

$$\phi_{YX}(-\tau) = \phi_{XY}(\tau)$$

Example



Example

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Suppose that $\phi_{XX}(\tau) = e^{-\alpha|\tau|}$. What is $\phi_{YY}(\tau)$?

We have

$$\Phi_{YY}(f) = |H(f)|^2 \Phi_{XX}(f)$$

where

$$|H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}$$

$$\Phi_{XX}(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Hence,

$$\Phi_{YY}(f) = \frac{1}{1 + (2\pi fRC)^2} \cdot \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Example

Do you remember *partial fractions*? Now you need them!
We write

$$\Phi_{YY}(f) = \frac{A}{\alpha^2 + (2\pi f)^2} + \frac{B}{1 + (2\pi f RC)^2}$$

and solve for A and B. We have

$$A(1 + (2\pi f RC)^2) + B(\alpha^2 + (2\pi f)^2) = 2\alpha$$

Clearly,

$$\begin{aligned} A + B\alpha^2 &= 2\alpha \\ A(2\pi f RC)^2 + B(2\pi f)^2 &= 0 \end{aligned}$$

From the second equation

$$A = -\frac{B}{(RC)^2} = -B\beta^2$$

where $\beta = 1/(RC)$.

Example

Then using the first equation

$$B = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Also,

$$A = -B\beta^2 = -\frac{2\alpha\beta^2}{\alpha^2 - \beta^2}$$

Finally,

$$\Phi_{YY}(f) = \frac{\beta^2}{\beta^2 - \alpha^2} \cdot \frac{2\alpha}{\alpha^2 + (2\pi f)^2} + \frac{\alpha\beta}{\alpha^2 - \beta^2} \cdot \frac{2\beta}{\beta^2 + (2\pi f)^2}$$

Now take inverse Fourier transforms to get

$$\phi_{YY}(\tau) = \frac{\beta^2}{\beta^2 - \alpha^2} \cdot e^{-\alpha|\tau|} + \frac{\alpha\beta}{\alpha^2 - \beta^2} \cdot e^{-\beta|\tau|}$$

Discrete-time Random Processes

Consider a wide-sense stationary discrete-time random process X_n that is input to a discrete-time linear time invariant filter with impulse response h_n.
The transfer function of the filter is

$$H(f) = \sum_{n=-\infty}^{\infty} h_n e^{-j2\pi fn}$$

The output of the filter is the convolution sum

$$Y_k = \sum_{n=-\infty}^{\infty} h_n X_{k-n}$$

It follows that the output mean is

$$\begin{aligned}\mu_Y = \mathrm{E}[Y_k] &= \sum_{n=-\infty}^{\infty} h_n \mathrm{E}[X_{k-n}] \\ &= \mu_X \sum_{n=-\infty}^{\infty} h_n \\ &= \mu_X H(0)\end{aligned}$$

Discrete-time Random Processes

The autocorrelation function of the output process is

$$\begin{aligned}\phi_{YY}(k) &= \mathrm{E}[Y_n Y_{n+k}] \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j \mathrm{E}[X_{n-i} h_j X_{n+k-j}] \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j \phi_{XX}(k-j+i)]\end{aligned}$$

By taking the discrete-time Fourier transform of $\phi_{YY}(k)$ and using the above relationship, we can obtain

$$\Phi_{YY}(f) = \Phi_{XX}(f)|H(f)|^2$$

Again, note in this case that $\Phi_{YY}(f)$ is periodic in f with a period of unity.