EE4601
Communication Systems

Week 5
Noise and Matched Filters

Error Probability with Binary Signaling
Thermal Noise

Thermal noise affects all communication receivers.

From fundamental physics (which we will not go into here) the power spectral density of thermal noise is

$$\Phi_{nn}(f) = \frac{h|f|}{2(e^{h|f|/kT} - 1)} \text{ watts/Hz}$$

where

$$h = 6.62 \times 10^{-34} \text{ Joules} = \text{Planck's constant}$$
$$k = 1.37 \times 10^{-23} \text{ Joules/degree} = \text{Boltzmann's constant}$$

Using the Taylor series expansion $$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$ gives

$$\Phi_{nn}(f) \approx \frac{h|f|}{2(1 + h|f|/kT - 1)}$$
$$= \frac{kT}{2} \text{ watts/Hz}$$
White Noise

Over a narrow bandwidth of frequencies the noise spectral density can be considered “flat”

\[ S_{nn}(f) = \frac{2kRT}{f^2} \]

\[ S_{nn}(f) \text{ vs } f \]

\[ N_0/2 \text{ vs } f \]

\[ -W \quad 0 \quad W \quad f \]

\[ 0 \quad 10^{12} \text{ Hz} \]

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White Noise

If we assume the bandwidth $W$ is infinite (idealization), then the autocorrelation of the noise is

$$\phi_{ww}(\tau) = \mathcal{F}^{-1}\{N_o/2\} = \frac{N_o}{2}\delta(\tau)$$

where we use a subscript ”w” to emphasize that the noise is white. Note that $w(t)$ is uncorrelated with $w(t + \tau)$ for any $\tau \neq 0$.

The noise power in bandwidth $W$ is

$$P_n = 2 \times W \times \frac{N_o}{2} = N_oW \text{ watts}$$
Filtered White Noise

If the input noise spectral density is $\Phi_{ww}(f) = N_0/2$, then the output noise spectral density is

$$\Phi_{nn}(f) = \frac{N_0}{2} |H(f)|^2$$

For example, consider the ideal low-pass filter

$$H(f) = \text{rect} \left( \frac{f}{2W} \right)$$

Then

$$\Phi_{nn}(f) = \frac{N_0}{2} \text{rect} \left( \frac{f}{2W} \right)$$
Filtered White Noise

The autocorrelation function of the ideal low-pass filtered noise is

\[ \phi_{nn}(\tau) = \frac{N_o}{2} 2W \text{sinc}(2W \tau) \]
\[ = N_o W \text{sinc}(2W \tau) \]

Observe that regularly spaced samples of \( n(t) \) taken \( 1/(2W) \) seconds apart are uncorrelated. Interesting!
Bandpass Filtered White Noise

\[
H(f) = \text{rect}\left(\frac{f - f_c}{2W}\right) + \text{rect}\left(\frac{f + f_c}{2W}\right)
\]

\[
\Phi_{nn}(f) = \frac{N_o}{2} \left[\text{rect}\left(\frac{f - f_c}{2W}\right) + \text{rect}\left(\frac{f + f_c}{2W}\right)\right]
\]

\[
\phi_{nn}(\tau) = \frac{N_o}{2} \cdot 2W \text{sinc}(2W\tau) \cdot 2\cos 2\pi f_c \tau
\]

\[
= 2N_o W \text{sinc}(2W\tau) \cdot \cos 2\pi f_c \tau
\]
Consider an arbitrary filter with transfer function $H(f)$. If the input to the filter is white noise with power spectral density $N_0/2$, then the noise power at the output of the filter is

$$N_{out} = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = N_0 \int_{0}^{\infty} |H(f)|^2 df$$

Next suppose that the same noise process is applied to an ideal low-pass filter with bandwidth $B$ and zero frequency response $H(0)$. The noise at the output of the filter is

$$N_{out} = N_0 BH^2(0)$$

Equating the above two equations give the noise equivalent bandwidth

$$B = \frac{\int_{0}^{\infty} |H(f)|^2 df}{H^2(0)}$$
Basic Problem

A pulse $g(t)$ is transmitted over a noisy channel, representing a “0” or “1”. The pulse is assumed to have duration $T$.

Given the knowledge of $g(t)$, how do we choose $h(t)$ to minimize the effects of noise?

$\Phi_{ww}(f) = \frac{N_0}{2}$

Given the knowledge of $g(t)$, how do we choose $h(t)$ to minimize the effects of noise?
Matched Filter

\[ y(t) = g_o(t) + n(t) \]

where

\[ g_o(t) = g(t) * h(t) \]
\[ n(t) = w(t) * h(t) \]

We wish to maximize the peak pulse signal-to-noise ratio

\[ \eta = \frac{|g_o(T)|^2}{E[n^2(T)]} = \frac{\text{instantaneous signal power}}{\text{average noise power}} \]

where \( T = \) sampling instant.

We have \( \Phi_{nn}(f) = |H(f)|^2 \Phi_{ww}(f) = \frac{N_o}{2}|H(f)|^2 \)

\[ E[n^2(T)] = \phi_{nn}(0) = \int_{-\infty}^{\infty} \Phi_{nn}(f) df = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \]
\[ g_o(T) = \int_{-\infty}^{\infty} G(f) H(f) e^{j2\pi f T} df \]

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Matched Filter

Then,
\[ \eta = \frac{\left| \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \]

Choose \( H(f) \) to maximize \( \eta \).

Apply the Schwartz inequality

\[ | \int_{-\infty}^{\infty} x(f)y(f)df|^2 \leq \int_{-\infty}^{\infty} |x(f)|^2 df \int_{-\infty}^{\infty} |y(f)|^2 df \]

with equality iff \( x(f) = ky^*(f) \), \( k \) - arbitrary scalar constant

Hence,
\[ \left| \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT} df \right|^2 \leq \int_{-\infty}^{\infty} |G(f)|^2 df \int_{-\infty}^{\infty} |H(f)|^2 df \]

and
\[ \eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df \]

Since the RHS does not depend on \( H(f) \), we maximize \( \eta \) by choosing

\[ H_{\text{opt}}(f) = kG^*(f)e^{-j2\pi fT} \leftrightarrow kg(T - t) = h_{\text{opt}}(t) \]
Matched Filter

This gives

\[ \eta_{\text{max}} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df = \frac{E}{N_0/2} \]

Recall Rayleigh’s energy theorem

\[ E = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df \]

Example: \( g(t) = AU_T(t) = AU(t) - AU(t-T) \)

Example graph:

- A
- g(t)
- T
- t

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Matched Filter

\[ h(t) = kg(T - t) \]

\[ g_o(t) = g(t) \ast h(t) \]
Binary Signaling

Antipodal signaling

'1' → \( g(t) \)

'0' → \( -g(t) \)

\[
\begin{align*}
\text{Assume } g(t) \text{ was sent, i.e., '1' was sent} \\
y(t) &= \int_0^T x(\alpha) h(t - \alpha) d\alpha \\
\text{Note} \\
\ h(t - \alpha) &= g(T - t + \alpha) \\
\ h(T - \alpha) &= g(\alpha)
\end{align*}
\]
Binary Signaling

\[ y(T) = \int_0^T x(\alpha) h(T - \alpha) d\alpha \]
\[ = \int_0^T [g(\alpha) + w(\alpha)] g(\alpha) d\alpha \]
\[ = \int_0^T g^2(\alpha) d\alpha + \int_0^T w(\alpha) g(\alpha) d\alpha \]
\[ = E + w = y \]

\( w \) is a Gaussian random variable with mean and variance

\[ E[w] = E \left[ \int_0^T w(\alpha) g(\alpha) d\alpha \right] = \int_0^T E[w(\alpha)] g(\alpha) d\alpha = 0 \]

\[ \sigma_w^2 = E[w^2] = E \left[ \int_0^T w(\alpha) g(\alpha) d\alpha \int_0^T w(\beta) g(\beta) d\beta \right] \]
\[ = \int_0^T \int_0^T E[w(\alpha) w(\beta)] g(\alpha) g(\beta) d\alpha d\beta \]
Binary Signaling

\[
\sigma_w^2 = \int_0^T \int_0^T \phi_{w_w}(\alpha - \beta)g(\alpha)g(\beta)d\alpha d\beta \\
= \frac{N_o}{2} \int_0^T \int_0^T \delta(\alpha - \beta)g(\alpha)g(\beta)d\alpha d\beta \\
= \frac{N_o}{2} \int_0^T g^2(\alpha)d\alpha = \frac{N_oE}{2}
\]

Therefore, given that ‘1’ was sent, \( y = y(T) \) has the conditional pdf

\[
f_{y\mid'1'}(y\mid'1') = \frac{1}{\sqrt{2\pi\sigma_w}} \exp\left\{-\frac{(y - E)^2}{2\sigma_w^2}\right\}, \quad \sigma_w^2 = \frac{N_oE}{2}
\]

Likewise, given that ‘0’ was sent, \( y = y(T) \) has the conditional pdf

\[
f_{y\mid'0'}(y\mid'0') = \frac{1}{\sqrt{2\pi\sigma_w}} \exp\left\{-\frac{(y + E)^2}{2\sigma_w^2}\right\}, \quad \sigma_w^2 = \frac{N_oE}{2}
\]
Probability of Error

\[ P_e = P_{e|1'} P(1') + P_{e|0'} P(0') \]
\[ = P(y < 0|1') P(1') + P(y > 0|0') P(0') \]
\[ = Q \left( \frac{E}{\sqrt{N_o E}} \right) \cdot \frac{1}{2} + Q \left( \frac{E}{\sqrt{N_o E}} \right) \cdot \frac{1}{2} = Q \left( \sqrt{\frac{2E}{N_o}} \right) \]

If \( P(1') \neq P(0') \), then \( \lambda = 0 \) does not yield the smallest \( P_e \).
Example: On-Off keying

Let ‘1’ → \( g(t) \)
‘0’ → 0 transmit nothing

As before we use a filter matched to \( g(t) \) and sample the output
If ‘1’ is sent then
\[
y = E + w \quad w \sim N(0, N_oE/2) \\
f_{y|1'} \sim N(E, N_oE/2)
\]
If ‘0’ is sent then
\[
y = 0 + w \quad w \sim N(0, N_oE/2) \\
f_{y|0'} \sim N(0, N_oE/2)
\]
On-Off Keying

For equally likely binary signals, the optimum slicer threshold (minizes $P_e$) is where the conditional pdfs cross. In this case $\lambda = E/2$.

\[
P_e = P_{e|1'}P(1') + P_{e|0'}P(0') \\
= P(y < E/2|1')P(1') + P(y > E/2|0')P(0') \\
= Q\left(\frac{E/2}{\sqrt{N_o E/2}}\right) \frac{1}{2} + Q\left(\frac{E/2}{\sqrt{N_o E/2}}\right) \frac{1}{2} \\
= Q\left(\frac{E/2}{\sqrt{N_o E/2}}\right) = Q\left(\frac{\sqrt{E}}{\sqrt{N_o}}\right)
\]

$\bar{E} = E/2$ - average bit energy

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On-Off Keying

\[ P_e = Q \left( \sqrt{\frac{E}{N_0}} \right) \text{ for On-Off keying} \]
\[ P_e = Q \left( \sqrt{\frac{2E}{N_0}} \right) \text{ for antipodal signals} \]