EE4601
Communication Systems

Week 6
Orthogonal Expansions
Basic Problem

Problem:

Suppose that we have a set of $M$ finite energy signals $S = \{s_1(t), s_2(t), \ldots, s_M(t)\}$, where each signal has a duration $T$ seconds.

Every $T$ seconds one of the waveforms from the set $S$ is selected for transmission over an AWGN channel. The transmitted waveform is

$$x(t) = \sum_n s_n(t - nT)$$

The received noise corrupted waveform is

$$r(t) = \sum_n s_n(t - nT) + n(t)$$

By observing $r(t)$ we wish to determine the time sequence of waveforms $\{s_n(t)\}$ that was transmitted. That is, in each $T$ second interval, we must determine which $s_i(t) \in S$ was transmitted.
Orthogonal Expansions

Consider a real valued signal $s(t)$ with finite energy $E_s$,

$$E_s = \int_{-\infty}^{\infty} s^2(t) dt$$

Suppose there exists a set of orthonormal functions $\{f_n(t)\}$, $n = 1, \ldots, N$. By orthonormal we mean

$$\int_{-\infty}^{\infty} f_n(t) f_k(t) dt = \delta_{kn} \quad \delta_{kn} = \begin{cases} 1 , & k = n \\ 0 , & k \neq n \end{cases}$$

We now approximate $s(t)$ as the weighted linear sum

$$\hat{s}(t) = \sum_{k=1}^{N} s_k f_k(t)$$

and wish to determine the $s_k, k = 1, \ldots, N$ to minimize the square error

$$\varepsilon = \int_{-\infty}^{\infty} (s(t) - \hat{s}(t))^2 dt$$

$$= \int_{-\infty}^{\infty} \left( s(t) - \sum_{k=1}^{N} s_k f_k(t) \right)^2 dt$$

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Orthogonal Expansions

To minimize the mean square error, we take the partial derivative with respect to each of the $s_k$ and set equal to zero, i.e., for the $n$th term we solve

$$\frac{\partial \varepsilon}{\partial s_n} = 2 \int_{-\infty}^{\infty} \left( s(t) - \sum_{k=1}^{N} s_k f_k(t) \right) f_n(t) dt = 0.$$ 

Using the orthonormal property of the basis functions, $s_n = \int_{-\infty}^{\infty} s(t) f_n(t) dt$ and

$$\varepsilon = \int_{-\infty}^{\infty} \left( s(t) - \sum_{k=1}^{N} s_k f_k(t) \right)^2 dt$$

$$= \int_{-\infty}^{\infty} s^2(t) dt - 2 \int_{-\infty}^{\infty} s(t) \sum_{k=1}^{N} s_k f_k(t) dt + \int_{-\infty}^{\infty} \sum_{k=1}^{N} s_k f_k(t) \sum_{\ell=1}^{N} s_{\ell} f_{\ell}(t) dt$$

$$= \int_{-\infty}^{\infty} s^2(t) dt - 2 \sum_{k=1}^{N} s_k \int_{-\infty}^{\infty} s(t) f_k(t) dt + \sum_{k=1}^{N} \sum_{\ell=1}^{N} s_k s_{\ell} \int_{-\infty}^{\infty} f_k(t) f_{\ell}(t) dt$$

$$= E_s - \sum_{k=1}^{N} \lambda_k^2$$

For a complete set of basis functions $\varepsilon = 0$. 

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Gram-Schmidt Orthonormalization

Suppose that we have a set of finite energy real signals \( \{s_i(t)\}, i = 1, \ldots, M \). We wish to obtain a complete set of orthonormal basis functions for the signal set. This can be done in 2 steps.

**Step 1**: Determine if the set of waveforms is linearly independent. If they are linearly dependent, then there exists a set of coefficients \( a_1, a_2, \ldots, a_M \), not all zero, such that

\[
a_1s_1(t) + a_2s_2(t) + \cdots + a_M s_M(t) = 0.
\]

Suppose, without loss of generality, that \( a_M \neq 0 \). If \( a_M = 0 \), then the signal set can be permuted so that \( a_M \neq 0 \). Then

\[
s_M(t) = -\left( \frac{a_1}{a_M}s_1(t) + \frac{a_2}{a_M}s_2(t) + \cdots + \frac{a_{M-1}}{a_M}s_{M-1}(t) \right).
\]

Next consider the reduced signal set \( \{s_i(t)\}_{i=1}^{M-1} \). If this set of waveforms is linearly dependent, then there exists another set of co-efficients \( \{b_i\}_{i=1}^{M-1} \), not all zero, such that

\[
b_1s_1(t) + b_2s_2(t) + \cdots + b_{M-1}s_{M-1}(t) = 0 .
\]
Gram-Schmidt Orthonormalization

We continue until a set \( \{s_i(t)\}_{i=1}^N \) of linearly independent waveforms is obtained. Note that \( N \leq M \) with equality if and only if the set of waveforms \( \{s_i(t)\}_{i=1}^M \) is linearly independent.

If \( N < M \), then the set of linearly independent waveforms \( \{s_i(t)\}_{i=1}^N \) is not unique, but any one will do.

**Step 2:** From the set \( \{s_i(t)\}_{i=1}^N \) construct the set of \( N \) orthonormal basis functions \( \{f_i(t)\}_{i=1}^N \) as follows. First, let

\[
f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}
\]

where \( E_1 \) is the energy in the waveform \( s_1(t) \), given by

\[
E_1 = \int_0^T s_1^2(t) dt
\]

Then

\[
s_1(t) = \sqrt{E_1}f_1(t) = s_{11}f_1(t)
\]

where \( s_{11} = \sqrt{E_1} \).
Gram-Schmidt Orthonormalization

Next, by using the waveform $s_2(t)$ we obtain

$$s_{21} = \int_0^T s_2(t)f_1(t)dt$$

along with the intermediate function

$$g_2(t) = s_2(t) - s_{21}f_1(t)$$

Note that $g_2(t)$ is orthogonal to $f_1(t)$.

The second basis function is

$$f_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T (g_2(t))^2 dt}} = \frac{s_2(t) - s_{21}f_1(t)}{\sqrt{E_2 - s_{21}^2}}$$
Gram-Schmidt Orthonormalization

Continuing in the above fashion, we define the $i$th intermediate function

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} f_j(t)$$

where

$$s_{ij} = \int_0^T s_i(t) f_j(t) \, dt$$

The set of functions

$$f_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T (g_i(t))^2}} \quad i = 1, 2, \ldots, N$$

is the required set of complete orthonormal basis functions.
Gram-Schmidt Orthonormalization

We can now write the signals as weighted linear combinations of the basis functions, i.e.,

\[
\begin{align*}
  s_1(t) &= s_{11} f_1(t) \\
  s_2(t) &= s_{21} f_1(t) + s_{22} f_2(t) \\
  s_3(t) &= s_{31} f_1(t) + s_{32} f_2(t) + f_{33} f_3(t) \\
  \vdots &= \vdots \\
  s_N(t) &= s_{N1} f_1(t) + \cdots + s_{NN} f_N(t)
\end{align*}
\]

For the remaining signals \(s_i(t), i = N + 1, \ldots, M\), we have

\[
    s_i(t) = \sum_{k=1}^{N} s_{ik} f_k(t)
\]

where

\[
    s_{ik} = \int_0^T s_i(t) f_k(t)\,dt
\]
Signal Vectors

It follows that the signal set $s_i(t), i = 1, \ldots, M$ can be expressed in terms of a set of signal vectors $s_i, i = 1, \ldots, M$ in an $N$-dimensional signal space, i.e.,

$$s_1(t) \leftrightarrow s_1 = (s_{11}, s_{12}, \ldots, s_{1N})$$
$$s_2(t) \leftrightarrow s_2 = (s_{21}, s_{22}, \ldots, s_{2N})$$
$$\vdots$$
$$s_M(t) \leftrightarrow s_M = (s_{M1}, s_{M2}, \ldots, s_{MN})$$

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Example

\[ s_1(t) \]

\[ s_2(t) \]

\[ s_3(t) \]

\[ s_4(t) \]
Example

Step 1: This signal set is not linearly independent because

\[ s_4(t) = s_1(t) + s_3(t) \]

Therefore, we will use \( s_1(t) \), \( s_2(t) \), and \( s_3(t) \) to obtain the complete orthonormal set of basis functions.

Step 2:

\( a) \)

\[ E_1 = \int_0^T s_1^2(t) dt = T/3 \]

\[ f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \sqrt{3/T}, & 0 \leq t \leq T/3 \\ 0, & \text{else} \end{cases} \]
Example

b)

\[ s_{21} = \int_0^T s_2(t) f_1(t) \, dt \]
\[ = \int_0^{T/3} \sqrt{3/T} \, dt = \sqrt{T/3} \]

\[ E_2 = \int_0^T s_2^2(t) \, dt = 2T/3 \]

\[ f_2(t) = \frac{s_2(t) - s_{21} f_1(t)}{\sqrt{E_2 - s_{21}^2}} \]
\[ = \begin{cases} \sqrt{3/T} & T/3 \leq t \leq 2T/3 \\ 0 & \text{else} \end{cases} \]
Example

c)

\[ s_{31} = \int_0^T s_3(t)f_1(t)dt = 0 \]
\[ s_{32} = \int_0^T s_3(t)f_2(t)dt = \int_{T/3}^{2T/3} \sqrt{3/T} dt = \sqrt{T/3} \]

\[ g_3(t) = s_3(t) - s_{31}f_1(t) - s_{32}f_2(t) = \begin{cases} 
1 & 2T/3 \leq t \leq T \\
0 & \text{else}
\end{cases} \]

\[ f_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}} = \begin{cases} 
\sqrt{3/T} & 2T/3 \leq t \leq T \\
0 & \text{else}
\end{cases} \]
Example

\[ f(t) = \begin{cases} \sqrt{3}T & \text{for } 0 \leq t < \frac{T}{3} \\ \sqrt{3}T & \text{for } \frac{T}{3} \leq t < \frac{2T}{3} \\ \sqrt{3}T & \text{for } \frac{2T}{3} \leq t < T \\ \end{cases} \]
Example

\[ s_1(t) \leftrightarrow s_1 = (\sqrt{T/3}, 0, 0) \]
\[ s_2(t) \leftrightarrow s_2 = (\sqrt{T/3}, \sqrt{T/3}, 0) \]
\[ s_3(t) \leftrightarrow s_3 = (0, \sqrt{T/3}, \sqrt{T/3}) \]
\[ s_4(t) \leftrightarrow s_4 = (\sqrt{T/3}, \sqrt{T/3}, \sqrt{T/3}) \]
Properties of Signal Vectors

Signal Energy:

\[ E = \int_0^T s^2(t) dt \]
\[ = \int_0^T \sum_{k=1}^N s_k f_k(t) \sum_{\ell=1}^N s_\ell f_\ell \, dt \]
\[ = \sum_{k=1}^N \sum_{\ell=1}^N s_k s_\ell \int_0^T f_k(t) f_\ell(t) dt \]
\[ = \sum_{k=1}^N s_k^2 \]
\[ \Delta \equiv \|s\|^2 \]

The energy in \( s(t) \) is just the squared length of its signal vector \( s \).
Properties of Signal Vectors

**Signal Correlation:** The correlation or "similarity" between two signals $s_j(t)$ and $s_k(t)$ is

$$\rho_{jk} = \frac{1}{\sqrt{E_j E_k}} \int_0^T s_j(t)s_k(t) dt$$

$$= \frac{1}{\sqrt{E_j E_k}} \int_0^T \sum_{n=1}^N s_{jn} f_n(t) \sum_{m=1}^N s_{km} f_m(t) dt$$

$$= \frac{1}{\sqrt{E_j E_k}} \sum_{n=1}^N \sum_{m=1}^N s_{jn} s_{km} \int_0^T f_n(t) f_m(t) dt$$

$$= \frac{1}{\sqrt{E_j E_k}} \sum_{n=1}^N s_{jn} s_{kn}$$

$$= \frac{s_j \cdot s_k}{\|s_j\| \|s_k\|}$$

Note that

$$\rho = \begin{cases} 
0 & \text{if } s_j(t) \text{ and } s_k(t) \text{ are orthogonal} \\
\pm 1 & \text{if } s_j(t) = \pm s_k(t)
\end{cases}$$
Properties of Signal Vectors

**Euclidean Distance:** The Euclidean distance between two signals $s_j(t)$ and $s_k(t)$ is

$$d_{jk} = \left\{ \int_0^T (s_j(t) - s_k(t))^2 dt \right\}^{1/2}$$

$$= \left\{ \int_0^T \left( \sum_{n=1}^N s_{jn} f_n(t) - \sum_{m=1}^N s_{km} f_m(t) \right)^2 dt \right\}^{1/2}$$

$$= \left\{ \sum_{n=1}^N (s_{jn} - s_{kn})^2 \right\}^{1/2}$$

$$= \left\{ \|s_j - s_k\|^2 \right\}^{1/2}$$

$$= \|s_j - s_k\|$$
Example

Consider the earlier example where

\[ s_1 = \left( \sqrt{\frac{T}{3}}, 0, 0 \right) \]
\[ s_2 = \left( \sqrt{\frac{T}{3}}, \sqrt{\frac{T}{3}}, 0 \right) \]
\[ s_3 = \left( 0, \sqrt{\frac{T}{3}}, \sqrt{\frac{T}{3}} \right) \]

We have \( E_1 = \|s_1\|^2 = \frac{T}{3}, E_2 = \|s_2\|^2 = \frac{2T}{3} \), and \( E_3 = \|s_3\|^2 = \frac{2T}{3} \).

The correlation between \( s_2(t) \) and \( s_3(t) \) is

\[ \rho_{23} = \frac{s_2 \cdot s_3}{\|s_2\| \|s_3\|} = \frac{T/3}{2T/3} = 0.5 \]

The Euclidean distance between \( s_1(t) \) and \( s_3(t) \) is

\[ d_{13} = \|s_1 - s_3\| = \sqrt{\frac{T}{3} + \frac{T}{3} + \frac{T}{3}} = \sqrt{T} \]