EE4061
Communication Systems

Lecture 18

Coherent Signal Detection
Suppose that we have $M$ signals $s_1(t), s_2(t), \ldots, s_M(t)$ that are defined over the time interval $0 \leq t \leq T$.

We select one of signals say $s_i(t)$ for transmission over an AWGN channel.

The received signal is

$$r(t) = \alpha s_i(t - t_o) + n(t)$$

where $n(t)$ is AWGN with power spectral density $N_o/2$ watts/Hz

Problem: By observing $r(t)$ determine which of the $M$ signals was (most likely) transmitted.

Assume that the receiver knows $\alpha$ and $t_o$ exactly. In this case, we can just assume that

$$r(t) = s_i(t) + n(t)$$
Correlation Detector

Any signal set \( \{s_1(t), s_2(t), \ldots, s_M(t)\} \) can be expressed in terms of a set of orthonormal basis functions \( \{f_1(t), f_2(t), \ldots, f_N(t)\} \) where \( N \) is the dimension of the signal space.

However, the basis functions do not span the noise space, i.e., the noise waveform \( n(t) \) cannot be represented exactly in terms of the \( N \) basis functions. We have

\[
    r(t) = \sum_{k=1}^{N} r_k f_k(t)
\]

where

\[
    r_k = \int_{0}^{T} r(t) f_k(t) dt = \int_{0}^{T} s_m(t) f_k(t) dt + \int_{0}^{T} n(t) f_k(t) dt = s_{mk} + n_k
\]

Hence, the projection of the received signal \( r(t) \) onto the signal space yields the received vector \( \mathbf{r} = (r_1, r_2, \ldots, r_N) \).
Correlation Detector

![Diagram of correlation detector](image)

\( r(t) \)

\( f_1(t) \)

\( f_2(t) \)

\( f_N(t) \)

\( t = T \)

\( r_1 \)

\( r_2 \)

\( r_{N-1} \)

\( r_N \)

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Noise Statistics

The noise components $n_k$ have mean

$$E[n_k] = \int_0^T E[n(t)] f_k(t) dt = 0$$

and covariance

$$E[n_j n_k] = \int_0^T \int_0^T E[n(t)n(s)] f_j(t)f_k(s) dt ds$$

$$= \frac{N_o}{2} \int_0^T \int_0^T \delta(t-s) f_j(t)f_k(s) dt ds$$

$$= \frac{N_o}{2} \int_0^T \int_0^T \delta(t-s) f_j(t)f_k(s) dt ds$$

$$= \frac{N_o}{2} \delta_{jk}$$

Therefore, the $r_k$ are independent Gaussian random variables with mean $s_{mk}$ and variance $\frac{N_o}{2}$. 
Joint Conditional Density

The vector \( \mathbf{r} \) has the joint conditional density function

\[
p(\mathbf{r}|\mathbf{s}_m) = \frac{1}{(\pi N_o)^{N/2}} \exp\left\{ -\frac{1}{N_o} \sum_{k=1}^{N} (r_k - s_{mk})^2 \right\}
\]

\[
= \frac{1}{(\pi N_o)^{N/2}} \exp\left\{ -\frac{1}{N_o} \| \mathbf{r} - \mathbf{s}_m \|^2 \right\}
\]

which is a multivariate Gaussian distribution.

Note that

\[
r(t) = \sum_{k=1}^{N} r_k f_k(t) + n'(t)
\]

where \( n'(t) \) is the remainder process

\[
n'(t) = n(t) - \sum_{k=1}^{N} n_k f_k(t)
\]
We have
\[
E[n'(t)r_k] = E[n'(t)s_{mk} + E[n'(t)n_k] \\
= E[n'(t)n_k] \\
= E \left[ \left( n(t) - \sum_{j=1}^{N} n_j f_j(t) \right) n_k \right] \\
= \int_{0}^{T} E[n(t)n(\tau)] f_k(\tau) d\tau - \sum_{j=1}^{N} E[n_k n_j] f_j(t) \\
= \frac{1}{2} N_o f_k(t) - \frac{1}{2} N_o f_k(t) = 0
\]

Hence, the vector \( r \) is uncorrelated with \( n'(t) \) and, therefore, \( n'(t) \) is irrelevant since it does not contain any information about \( r \).

This is Wozencraft’s irrelevance theorem which is certainly not irrelevant!
Suppose that we filter the received signal $r(t)$ with a bank of matched filters having the impulse responses

$$h_k(t) = f_k(T - t) \quad , \quad 0 \leq t \leq T$$

and sample the filter outputs at time $t = T$.

The filter outputs are

$$y_k(t) = \int_0^t r(\tau) h_k(t - \tau) d\tau$$

$$= \int_0^t r(\tau) f_k(T - t + \tau) d\tau$$

$$y_k(T) = \int_0^T r(\tau) f_k(\tau) d\tau$$

Note that $y_k = r_k$, i.e., the matched filter outputs are identical to the correlator outputs.
Matched Filter Receiver

\[ r(t) \]

\[ f_1(T-t) \rightarrow y_1 \]
\[ f_2(T-t) \rightarrow y_2 \]
\[ \vdots \]
\[ f_N(T-t) \rightarrow y_N \]

\[ t=T \]

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Minimum Distance Decisions

With minimum distance decisions, the receiver first calculates the vector $\mathbf{r}$. The receiver then decides in favour of the signal point $\mathbf{s}_i$ that is closest in Euclidean distance or squared Euclidean distance to the received vector $\mathbf{r}$.

The minimum distance decision rule is

$$\hat{s} = \arg \min_{\mathbf{s}_i} \| \mathbf{r} - \mathbf{s}_i \|^2$$

Since, the vector $\mathbf{r}$ has the joint conditional density function

$$p(\mathbf{r}|s_m) = \frac{1}{(\pi N_0)^{N/2}} \exp \left\{ -\frac{1}{N_0} \| \mathbf{r} - \mathbf{s}_m \|^2 \right\}$$

the choice of $\mathbf{s}_i$ that minimizes $\| \mathbf{r} - \mathbf{s}_i \|^2$ also maximizes the likelihood $p(\mathbf{r}|s_m)$. This is also called a maximum likelihood receiver.