EE6604
Personal & Mobile Communications

Week 11

Continuous Phase Modulation
Continuous Phase Modulation (CPM)

- The CPM bandpass signal is

\[
s(t) = \text{Re}\left\{ Ae^{j\phi(t)}e^{j2\pi f_c t} \right\} \\
\]

\[
= A \cos \left( 2\pi f_c t + \phi(t) \right)
\]

(1)

where the “excess phase” is

\[
\phi(t) = 2\pi h \int_0^t \sum_{k=0}^{\infty} x_k h_f(\tau - kT) d\tau
\]

- h is the modulation index
- \( x_n \in \{\pm 1, \pm 3, \ldots, \pm (M - 1)\} \) are the M-ary data symbols
- \( h_f(t) \) is the “frequency shaping pulse” of duration \( LT \), that is zero for \( t < 0 \) and \( t > LT \), and normalized to have an area equal to 1/2. Full response CPM has \( L = 1 \), while partial response CPM has \( L > 1 \).

- The instantaneous frequency deviation from the carrier is

\[
f_{\text{dev}}(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = h \sum_{k=0}^{\infty} x_k h_f(t - kT)
\]
## Frequency Shaping Pulses

<table>
<thead>
<tr>
<th>pulse type</th>
<th>$h_f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$-rectangular (LREC)</td>
<td>$\frac{1}{2LT}u_{LT}(t)$</td>
</tr>
<tr>
<td>$L$-raised cosine (LRC)</td>
<td>$\frac{1}{2LT} \left[ 1 - \cos \left( \frac{2\pi t}{LT} \right) \right] u_{LT}(t)$</td>
</tr>
<tr>
<td>$L$-half sinusoid (LHS)</td>
<td>$\frac{\pi}{4LT} \sin(\pi t/LT)u_{LT}(t)$</td>
</tr>
<tr>
<td>$L$-triangular (LTR)</td>
<td>$\frac{1}{LT} \left( 1 - \frac{</td>
</tr>
</tbody>
</table>
Excess Phase and Tilted Phase

- During the time interval \( nT \leq t \leq (n + 1)T \), the excess phase \( \phi(t) \) is

\[
\phi(t) = 2\pi h \sum_{k=0}^{n} x_k \beta(t - kT).
\]

where the “phase shaping pulse” is

\[
\beta(t) = \begin{cases}
0 & , \ t < 0 \\
\int_{0}^{t} h_f(\tau)d\tau & , \ 0 \leq t \leq LT \\
1/2 & , \ t \geq LT
\end{cases}
\]

- For the case of full response CPM \((L = 1)\), during the time interval \( nT \leq t \leq (n + 1)T \) the excess phase is

\[
\phi(t) = \pi h \sum_{k=0}^{n-1} x_k + 2\pi h x_n \beta(t - nT)
\]

- During the time interval \( nT \leq t \leq (n + 1)T \), the CPM “tilted phase” is

\[
\psi(t) = \pi h \sum_{k=0}^{n-1} x_k + 2\pi h x_n \beta(t - nT) + \pi h (M - 1)t/T
\]

\[
= \phi(t) + \pi h (M - 1)t/T
\]

- Note that \( s(t) \) can be generated by replacing \( \phi(t) \) with \( \psi(t) \) and \( f_c \) by \( f_c - h(M - 1)t/2T \) in (1).
Continuous Phase Frequency Shift Keying (CPFSK)

- Continuous phase frequency shift keying (CPFSK) is a special type of CPM that uses the full response REC shaping function

\[ h_f(t) = \frac{1}{2T}u_T(t) = \frac{1}{2T}(u(t) - u(t-T)) \]

As a result

\[ \beta(t) = \begin{cases} 
0 & , t < 0 \\
\frac{t}{2T} & , 0 \leq t \leq T \\
\frac{1}{2} & , t \geq T 
\end{cases} \]

- Since the frequency shaping function is rectangular, the phase shaping pulse contains a linear ramp and the CPFSK excess phase trajectories are linear.
Phase tree of binary CPFSK.
Phase-state Diagrams

Phase-state diagram of CPM with $h = 1/4$. 
Minimum Shift Keying (MSK)

- MSK is a special case of CPFSK, where the modulation index \( h = \frac{1}{2} \) is used.
- The phase shaping pulse is

\[
\beta(t) = \begin{cases} 
0 & , \quad t < 0 \\
t/2T & , \quad 0 \leq t \leq T \\
1/2 & , \quad t \geq T 
\end{cases}
\]

- The MSK bandpass waveform is

\[
s(t) = A \cos \left( 2\pi f_c t + \frac{\pi}{2} \sum_{k=0}^{n-1} x_k + \frac{t - nT}{2T} \pi x_n \right), \quad nT \leq t \leq (n + 1)T
\]

- The excess phase on the interval \( nT \leq t \leq (n + 1)T \) is

\[
\phi(t) = \frac{\pi}{2} \sum_{k=0}^{n-1} x_k + \frac{t - nT}{2T} \pi x_n
\]

- The tilted phase on the interval \( nT \leq t \leq (n + 1)T \) is

\[
\psi(t) = \phi(t) + \frac{\pi t}{2T}
\]

- Combining the above two equations, we have

\[
\psi((n + 1)T) = \psi(nT) + \frac{\pi}{2}(1 + x_n)
\]
Excess Phase and Tilted Phase for Minimum Shift Keying (MSK)

Example: MSK \( h = \frac{1}{2} \)

<table>
<thead>
<tr>
<th>t/T</th>
<th>Excess Phase</th>
<th>Tilted Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>1</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>4</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>+1</td>
<td>-1</td>
</tr>
</tbody>
</table>

4-states

2-states
Phase state diagram for MSK signals.
Linearized Representation of MSK

An interesting representation for MSK waveforms can be obtained by using Laurent’s decomposition to express the MSK complex envelope in the quadrature form

\[ \tilde{s}(t) = A \sum_n b(t - 2nT, x_n) , \]

where

\[ b(t, x_n) = \hat{x}_{2n+1} h_a(t - T) + j \hat{x}_{2n} h_a(t) \]

and where \( x_n = (\hat{x}_{2n+1}, \hat{x}_{2n}) , \)

\begin{align*}
\hat{x}_{2n} &= \hat{x}_{2n-1} x_{2n} \\
\hat{x}_{2n+1} &= -\hat{x}_{2n} x_{2n+1} \\
\hat{x}_{-1} &= 1
\end{align*}

and

\[ h_a(t) = \sin \left( \frac{\pi t}{2T} \right) u_{2T}(t) . \]

The sequences, \( \{\hat{x}_{2n}\} \) and \( \{\hat{x}_{2n+1}\} \), are independent binary symbol sequences taking on elements from the set \( \{-1, +1\} \).

The symbols \( \hat{x}_{2n} \) and \( \hat{x}_{2n+1} \) are transmitted on the quadrature branches with a half-sinusoid (HS) amplitude shaping pulse of duration \( 2T \) seconds and an offset of \( T \) seconds.
Gaussian MSK (GMSK)

- With MSK the modulating signal is
  \[ x(t) = \frac{1}{2T} \sum_{n=-\infty}^{\infty} x_n u_T(t - nT) \]

- The bandwidth of \( \tilde{s}(t) \) depends on the bandwidth of \( x(t) \) and the modulation index \( h \). For GMSK \( h = 1/2 \).

- We filter \( x(t) \) with a low-pass filter to remove high frequency content prior to modulation, i.e., we use the filtered pulse \( g(t) = x(t) * h(t) \).

- For GMSK, the low-pass filter transfer function is
  \[ H(f) = \exp \left\{ - \left( \frac{f}{B} \right)^2 \frac{\ln 2}{2} \right\} \]
  where \( B \) is the 3 dB filter bandwidth.

Gaussian Pre-modulation filtered MSK (GMSK).
• A rectangular pulse \( \text{rect}(t/T) = u_T(t+T/2) \) transmitted through this Gaussian low-pass filter yields the GMSK frequency shaping pulse

\[
h_f(t) = \frac{1}{2T} \left[ \frac{2\pi}{\ln 2} (BT) \right]^{\frac{t}{T+1/2}} \exp \left\{ -\frac{2\pi^2 (BT)^2 x^2}{\ln 2} \right\} dx
\]

\[
= \frac{1}{2T} \left[ Q \left( \frac{t/T - 1/2}{\sigma} \right) - Q \left( \frac{t/T + 1/2}{\sigma} \right) \right]
\]

where

\[
Q(\alpha) = \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
\]

\[
\sigma^2 = \frac{\ln 2}{4\pi^2 (BT)^2}.
\]

• The total pulse area is \( \int_{-\infty}^{\infty} h_f(t) dt = 1/2 \) and, therefore, the total contribution to the excess phase for each data symbol is \( \pm \pi/2 \) radians.
GMSK frequency shaping pulse for various normalized filter bandwidths $BT$. 
• The GMSK phase shaping pulse is

\[ \beta(t) = \int_{-\infty}^{t} h_f(t) dt = \frac{1}{2} \left( G \left( \frac{t}{T} + \frac{1}{2} \right) - G \left( \frac{t}{T} - \frac{1}{2} \right) \right) \]

where

\[ G(x) = x \Phi \left( \frac{x}{\sigma} \right) + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \]

and

\[ \Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \]

• Observe that \( \beta(\infty) = 1/2 \) and, therefore, the total contribution to the excess phase for each data symbol remains at \( \pm \pi/2 \) as mentioned earlier.
GMSK phase shaping pulse for various normalized filter bandwidths $BT$. 
The excess phase change over the interval from $-T/2$ to $T/2$ is

$$\phi(T/2) - \phi(-T/2) = x_0\beta_0(T) + \sum_{n=-\infty \atop n\neq 0}^{\infty} x_n\beta_n(T)$$

where

$$\beta_n(T) = \int_{-T/2-nT}^{T/2-nT} h_f(\nu) \, d\nu.$$ 

and

$$h_f(t) = \frac{1}{2T} \left[ Q \left( \frac{t/T - 1/2}{\sigma} \right) - Q \left( \frac{t/T + 1/2}{\sigma} \right) \right]$$

The first term, $x_0\beta_0(T)$, is the desired term, and the second term, $\sum_{n=-\infty \atop n\neq 0}^{\infty} x_n\beta_n(T)$, is the intersymbol interference (ISI) introduced by the Gaussian low-pass filter.

Conclusion: GMSK trades off power efficiency (due to the induced ISI) for a greatly improved bandwidth efficiency.

- the loss in power efficiency can be recovered by using an equalizer in the receiver to mitigate the induced ISI.
Power spectral density of GMSK with various normalized filter bandwidths $BT$. 
Linearized Gaussian Minimum Shift Keying (LGMSK)

• Laurent showed that any binary partial response CPM signal can be represented exactly as a linear combination of $2^{L-1}$ partial-response pulse amplitude modulated (PAM) signals, viz.,

$$\tilde{s}(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{2^{L-1}-1} e^{j\pi h \alpha_{n,p}} c_p(t - nT),$$

where

$$c_p(t) = c(t) \prod_{n=1}^{L-1} c(t + (n + L\varepsilon_{n,p})T),$$

$$\alpha_{n,p} = \sum_{m=0}^{n} x_m - \sum_{m=1}^{L-1} x_{n-m}\varepsilon_{m,p},$$

and $\varepsilon_{n,p} \in \{0, 1\}$ are the coefficients of the binary representation of the index $p$, i.e.,

$$p = \varepsilon_{0,p} + 2\varepsilon_{1,p} + \cdots + 2^{L-2}\varepsilon_{L-2,p}.$$

• The basic signal pulse $c(t)$ is

$$c(t) = \begin{cases} 
\frac{\sin(2\pi h \beta(t))}{\sin(\pi h)} & , \quad 0 \leq t < LT \\
\frac{\sin(\pi h - 2\pi h \beta(t-LT))}{\sin(\pi h)} & , \quad LT \leq t < 2LT \\
0 & , \quad \text{otherwise}
\end{cases}$$

where $\beta(t)$ is the CPM phase shaping function.
Linearized Gaussian Minimum Shift Keying (LGMSK)

• Note that the GMSK frequency shaping pulse spans $L = 3$ to $L = 4$ symbol periods for practical values of $BT$.

• Often the pulse $c_0(t)$ contains most of the signal energy, so the $p = 0$ term in can provide a good approximation to the CPM signal. Numerical analysis can show that the pulse $c_0(t)$ contains 99.83% of the energy and, therefore, we can derive a linearized GMSK waveform by using only $c_0(t)$ and neglecting the other pulses.

• This yields the waveform

$$\tilde{s}(t) = \sum_{n=0}^{\infty} e^{j\pi h \alpha_{n,0}} c_0(t - nT),$$

where, with $L = 4$,

$$c_0(t) = \prod_{n=0}^{3} c(t + nT),$$

$$\alpha_{n,0} = \sum_{m=0}^{n} x_m$$

• Since the GMSK phase shaping pulse is non-causal, when evaluating $c(t)$ we use the truncated and time shifted GMSK phase shaping pulse

$$\hat{\beta}(t) = \beta(t - 2T)$$

with $L = 4$ as shown previously.
LGMSK amplitude shaping pulse for various normalized premodulation filter bandwidths $BT$. 
Linearized Gaussian Minimum Shift Keying (LGMSK)

• For $h = 1/2$ used in GMSK,

\[ a_{n,0} = e^{j\frac{\pi}{2}\alpha_{n,0}} \in \{ \pm 1, \pm j \} , \]

and it follows that

\[ \tilde{s}(t) = A \sum_n \left( \hat{x}_{2n+1} c_0(t - 2nT - T) + j\hat{x}_{2n} c_0(t - 2nT) \right) \]

where

\[ \hat{x}_{2n} = \hat{x}_{2n-1}x_{2n} \]
\[ \hat{x}_{2n+1} = -\hat{x}_{2n}x_{2n+1} \]
\[ \hat{x}_{-1} = 1 \]

• This is the same as the OQPSK representation for MSK except that the half-sinusoid amplitude pulse shaping function is replaced with the LGMSK amplitude pulse shaping function.

• Note that the LGMSK pulse has length of approximately $3T$ to $4T$, while the pulses on the quadrature branches are transmitted every $2T$ seconds. Therefore, the LGMSK pulse introduces ISI, but this can be corrected with an equalizer in the receiver.