EE6604
Personal & Mobile Communications

Week 15

Multi-carrier Techniques
OFDM IMPAIRMENTS

- AWGN
- Timing offset *
- Doppler
- Carrier frequency offset *
- Intersymbol interference (ISI) *
- Large Peak-to-Average Power Ratio (PAPR)
FFT-based OFDM Transmitter
FFT-based OFDM Receiver
Performance in AWGN

- Suppose that the discrete-time OFDM time-domain sequence with a cyclic suffix, \( X^g_n = \{X^g_{n,m}\}_{m=0}^{N+G-1} \), is passed through a balanced pair of digital-to-analog converters (DACs), and the resulting complex envelope is transmitted over a quasi-static flat fading channel with complex gain \( g \).

- The receiver uses a quadrature demodulator to extract the received complex envelope \( \tilde{r}(t) = \tilde{r}_I(t) + j\tilde{r}_Q(t) \).

- Suppose that the quadrature components \( \tilde{r}_I(t) \) and \( \tilde{r}_Q(t) \) are each passed through an ideal anti-aliasing filter (ideal low-pass filter) having a cutoff frequency \( 1/(2T^g_s) \) followed by an analog-to-digital converter (ADC)

- This produces the received complex-valued sample sequence \( R^g_n = \{R^g_{n,m}\}_{m=0}^{N+G-1} \), where

\[
R^g_{n,m} = gX^g_{n,m} + \tilde{n}_{n,m},
\]

\( g = \alpha e^{j\phi} \) is the complex channel gain, and the \( \tilde{n}_{n,m} \) are the complex-valued Gaussian noise samples.

- For an ideal anti-aliasing filter having a cutoff frequency \( 1/(2T^g_s) \), the \( \tilde{n}_{n,m} \) are independent zero-mean complex Gaussian random variables with variance

\[
\sigma^2 = \frac{1}{2}E[|\tilde{n}_{n,m}|^2] = N_0/T^g_s, \text{ where } T^g_s = NT_s/(N + G).
\]
Performance in AWGN

- Assuming a cyclic suffix, the receiver first removes the guard interval according to
  \[ R_{n,m} = R_{n,G+(m-G)N}^g, \quad 0 \leq m \leq N-1, \]
  where \((m)_N\) is the residue of \(m\) modulo \(N\). Demodulation is then performed by computing the FFT on the block \(R_n = \{R_{n,m}\}_{m=0}^{N-1}\) to yield the vector \(z_n = \{z_{n,k}\}_{k=0}^{N-1}\) of \(N\) decision variables

  \[ z_{n,k} = \frac{1}{N} \sum_{m=0}^{N-1} R_{n,m} e^{-j2\pi km/N} = gAx_{n,k} + \nu_{n,k}, \quad k = 0, \ldots, N-1, \]
  where \(A = \sqrt{2E_h/T}, \quad T = (N + G)T_s^g\), and the noise terms are given by

  \[ \nu_{n,k} = \frac{1}{N} \sum_{m=0}^{N-1} \tilde{n}_{n,m} e^{-j2\pi km/N}, \quad k = 0, \ldots, N-1. \]

- It can be shown that the \(\nu_{n,k}\) are zero mean complex Gaussian random variables with covariance

  \[ \phi_{j,k} = \frac{1}{2} \mathbb{E}[\nu_{n,j}\nu_{n,k}^*] = \frac{N_o}{NT_s^g} \delta_{jk}. \]

  Hence, the \(z_{n,k}\) are independent Gaussian random variables with mean \(g\sqrt{2E_h/T}x_{n,k}\) and variance \(N_o/NT_s^g\). 


Performance in AWGN

- To be consistent with our earlier results for PSK and QAM signals, we can multiply the $z_{n,k}$ for convenience by the scalar $\sqrt{NT_s}$. Such scaling gives

$$\tilde{z}_{n,k} = g\sqrt{2E_hN/(N+G)}x_{n,k} + \tilde{\nu}_{n,k},$$

where the $\tilde{\nu}_{n,k}$ are i.i.d. zero-mean Gaussian random variables with variance $N_0$.

- Notice that $\sqrt{2E_hN/(N+G)}x_{n,k} = \tilde{s}_{n,k}$ is equal to the complex signal vector that is transmitted on the $i$th sub-carrier, where the term $N/(N+G)$ represents the loss in effective symbol energy due to the insertion of the cyclic guard interval.

- For each of the $\tilde{z}_{n,k}$, the receiver decides in favor of the signal vector $\tilde{s}_{n,k}$ that minimizes the squared Euclidean distance

$$\mu(\tilde{s}_{n,k}) = \|\tilde{z}_{n,k} - g\tilde{s}_{n,k}\|^2, \quad k = 0, \ldots, N - 1.$$

- It is apparent that the probability of symbol error is identical to that achieved with independent modulation on each of the sub-carriers. This is expected, because the sub-carriers are mutually orthogonal in time.
Timing Errors

• Timing offset smaller than the guard interval results in a phase shift

• Consider an OFDM waveform with cyclic prefix on an ideal channel such that $R_n = X_n$.

• Suppose that the receiver has a timing offset, $\ell$, that lies in the guard interval. Then instead of taking an FFT on the block $\{R_n\}_{n=0}^{N-1}$, the receiver takes an FFT on the block $\{\hat{R}_n\}_{n=0}^{N-1}$ instead, where $\hat{R}_n = R_{(n-\ell)N}$, where $(n)_N$ is the residue of $n$ modulo $N$. 
**Timing Errors**

- The FFT output becomes

\[
\hat{Z}_k = \frac{1}{N} \sum_{n=0}^{N-1} \hat{R}_n e^{-j \frac{2\pi kn}{N}} \\
= \frac{1}{N} \sum_{n=0}^{N-1} R_{(n-\ell)N} e^{-j \frac{2\pi kn}{N}} \\
= \frac{1}{N} \sum_{m=-\ell}^{N-\ell-1} R_m e^{-j \frac{2\pi k(m+\ell)}{N}} \\
= \left( \frac{1}{N} \sum_{m=0}^{N-1} R_m e^{-j \frac{2\pi km}{N}} \right) e^{-j \frac{2\pi k\ell}{N}} \\
= Z_k e^{-j \frac{2\pi k\ell}{N}}, \quad k = 0, \ldots, N-1.
\]

- Timing offset smaller than the guard interval results in a phase shift on each sub-carrier.
  - The effects of timing offset are easy to remove with adaptive channel estimation.

- If the timing offset is greater than the guard interval, additional interference is generated.
  - Best solution is to choose sufficient guard interval and make sure the timing offset lies in the guard interval.
Carrier Frequency Offset

- Carrier frequency offset also causes interchannel interference (ICI).
- The transmitted complex envelope with carrier frequency offset $\Delta f$ is
  \[
  \tilde{s}(t) = A \sum_{n=0}^{N-1} x_n \exp \left\{ j2\pi \left( \frac{n}{NT_s} + \Delta f \right) t \right\} u_T(t)
  \]
- The corresponding IFFT coefficients are
  \[
  X_k = A \sum_{n=0}^{N-1} x_n \exp \left\{ j2\pi \left( \frac{nk}{N} + k\Delta f T_s \right) \right\} , \quad k = 0, 1, \ldots, N + G - 1
  \]
- The demodulated sequence (in absence of noise) can be written as
  \[
  Z_l = \text{FFT}\{X_n\} = \eta x_l + c_l
  \]
  where
  \[
  \eta = A \left\{ \frac{\sin (\pi N \Delta f T_s)}{N \sin (\pi \Delta f T_s)} \right\} e^{j\pi(N-1)\Delta f T_s}
  \]
  and
  \[
  c_l = A \sum_{n \neq l} x_n H(n, l)
  \]
is the random ICI term, where
  \[
  H(m, l) = \left\{ \frac{\sin (\pi (m - l + N \Delta f T_s))}{N \sin (\pi (m - l + N \Delta f T_s) / N)} \right\} e^{j\pi(N-1)(m-l+N\Delta f T_s)}
  \]
Effect of Carrier Frequency Offset

- Assume that the receiver can determine \( \pi (N - 1)T_s \Delta f = \arg(\eta_l) \), the demodulated sequence in the absence of noise can be written as

\[
\hat{Z}_l = Z_l e^{-j\pi (N-1)T_s \Delta f} = \hat{\eta}_l x_l + \hat{c}_l
\]

where

\[
\hat{\eta}_l = A \left\{ \frac{\sin (\pi NT_s \Delta f)}{N \sin(\pi T_s \Delta f)} \right\}
\]

and

\[
\hat{c}_l = A \sum_{n \neq l} x_n \hat{H}(n, l)
\]

is the random ICI term, where

\[
\hat{H}(n, l) = \left\{ \frac{\sin \left[ \pi \left( n - l - \sin NT_s \Delta f \right) \right]}{\pi \left( n - l - NT_s \Delta f \right)} \right\} e^{j\pi (n-l)}
\]

- Observe that carrier frequency offset has two effects

1. It reduces the useful signal energy by the factor

\[
10 \log_{10} \left\{ \frac{\sin \left( \pi NT_s \Delta f \right)}{N \sin(\pi T_s \Delta f)} \right\}^2
\]

2. Introduces an additional additive noise term \( \hat{c}_l \)
Combating ISI with OFDM

• Suppose that the IFFT output vector \( X_n = \{X_{n,m}\}_{m=0}^{N-1} \) is appended with a cyclic suffix to yield the vector \( X^g_n = \{X^g_{n,m}\}_{m=0}^{N+G-1} \), where

\[
X^g_{n,m} = X_{n,(m)N} = A \sum_{k=0}^{N-1} x_{n,k} e^{\frac{j2\pi km}{N}}, \quad m = 0, 1, \ldots, N + G - 1,
\]

\( G \) is the length of the guard interval in samples, and \((m)_N\) is the residue of \( m \) modulo \( N \). To maintain the data rate \( R_s = 1/T_s \), the DAC in the transmitter is clocked with rate \( R_s^g = \frac{N+G}{N} R_s \), due to the insertion of the cyclic guard interval.

• Consider a time-invariant ISI channel with impulse response \( g(t) \). The combination of the DAC, waveform channel \( g(t) \), anti aliasing filter, and DAC yields an overall discrete-time channel with sampled impulse response \( g = \{g_m\}_{m=0}^L \), where \( L \) is the length of the discrete-time channel impulse response.

• The discrete-time linear convolution of the transmitted sequence \( \{X^g_n\} \) with the discrete-time channel produces the discrete-time received sequence \( \{R^g_{n,m}\} \), where

\[
R^g_{n,m} = \begin{cases} 
\sum_{i=0}^m g_i X^g_{n,m-i} + \sum_{i=m+1}^L g_i X^g_{n-1,N+G+m-i} + \tilde{n}_{n,m}, & 0 \leq m < L \\
\sum_{i=0}^L g_i X^g_{n,m-i} + \tilde{n}_{n,m}, & L \leq m \leq N + G - 1
\end{cases}
\]
Removal of Guard Interval

- To remove the ISI introduced by the channel, the first $G$ received samples $\{R^g_{n,m}\}_{m=0}^{G-1}$ are discarded and replaced with the last $G$ received samples $\{R^g_{n,m}\}_{m=N}^{N+G-1}$.

- If the length of the guard interval satisfies $G \geq L$, then we obtain the received sequence

$$R_{n,m} = R^g_{n,G+(m-G)N}$$

$$= \sum_{i=0}^{L} g_i X_{n,(m-i)N} + \tilde{n}_{n,(m-i)N}, \quad 0 \leq m \leq N - 1.$$

- Note that the first term represents a circular convolution of the transmitted sequence $X_n = \{X_{n,m}\}$ with the discrete-time channel $g = \{g_m\}_{m=0}^{L}$.

Removal of ISI using a cyclic suffix
The OFDM baseband demodulator computes the DFT of the vector \( R_n \). This yields the output vector

\[
    z_{n,i} = \frac{1}{N} \sum_{m=0}^{N-1} R_{n,m} e^{-j \frac{2\pi m i}{N}}
    = T_i \hat{A} x_{n,i} + \nu_{n,i}, \quad 0 \leq i \leq N - 1,
\]

where

\[
    T_i = \sum_{m=0}^{L} g_m e^{-j \frac{2\pi m i}{N}}
\]

and the noise samples \( \{\nu_{n,i}\} \) are i.i.d with zero-mean and variance \( N_o/(NT_s^g) \).

Note that \( T = \{T_i\}^{N-1}_{i=0} \) is the DFT of the zero padded sequence \( g = \{g_m\}^{N-1}_{m=0} \) and is equal to the sampled frequency response of the channel.

To be consistent with our earlier results, we can multiply the \( z_{n,i} \) for convenience by the scalar \( \sqrt{NT_s^g} \), giving

\[
    \tilde{z}_{n,i} = T_i \hat{A} x_{n,i} + \tilde{\nu}_{n,i} \quad i = 0, \ldots, N - 1,
\]

where \( \hat{A} = \sqrt{2E_hN/(N + G)} \) and the \( \tilde{\nu}_{n,i} \) are i.i.d. zero-mean Gaussian random variables with variance \( N_o \).

Observe that each \( \tilde{z}_{n,i} \) depends only on the corresponding data symbol \( x_{n,i} \) and, therefore, the ISI has been completely removed.

Once again, for each of the \( \tilde{z}_{n,i} \), the receiver decides in favor of the signal vector \( \tilde{s}_m \) that minimizes the squared Euclidean distance

\[
    \mu(\tilde{s}_m) = \|\tilde{z}_{n,i} - T_i \hat{A} x_{n,i}\|^2.
\]
OFDMA

- OFDMA achieves multiple access by assigning different users disjoint sets of sub-carriers.

- Assume that there are a total of $M$ sub-carriers that are evenly distributed among $Q$ users, such that each user is allocated $N = M/Q$ sub-carriers. The overall sub-carriers are labeled with indices from 0 to $M - 1$, while the $N$ sub-carriers allocated to the $j$th MS have indices that belong to the set $T_j$. Clearly, the sets $T_j$ must be disjoint such that each sub-carrier is assigned to at most one MS.

- The sub-carrier allocation can be performed by extending the $n$th data vector for the $j$th MS, denoted, by $a_{j,n}$ with the insertion of $M - N$ zeros in the sub-carriers belonging the set $\bar{T}_j$ which is the complement of $T_j$, i.e.,

$$x_{j,n,i} = \begin{cases} a_{j,n,i} , & \text{if } i \in T_j \\ 0 , & \text{otherwise} \end{cases},$$

where $a_{j,n,i}$ is the data symbol transmitted to the $j$th MS in block $n$ on the $i$th sub-carrier.
OFDMA - Forward Link Transmitter

- On the forward link, the vectors \( x_{j,n} = \{x_{j,n,i}\}_{i=0}^{M-1} \) are summed up to produce the \( n \)th data block

\[
x_n = \sum_{j=1}^{Q} x_{j,n}
\]

that is subsequently applied to an \( M \)-point IDFT to produce the length-\( M \) time-domain sequence \( X_n \).

- After the IDFT, a length-\( G \) guard interval is appended to each block in the form of a cyclic prefix or cyclic suffix, to yield the transmitted time-domain sequence \( X^g_n \).

- In the case of a cyclic prefix, the last \( G \) symbols of the sequence \( X_n = \{X_{n,m}, \ m = 0, \ldots, M-1\} \) are copied and appended to the beginning of \( X_n \). The transmitted time-domain sequence for the \( n \)th block with guard interval, denoted as \( X^g_n \), is

\[
X^g_n = \{X_{n,(m)_M}, \ m = -G, -G + 1, \ldots, -1, 0, 1, \ldots, M-1\}
\]

where \((m)_M\) is the residue of \( m \) modulo-\( M \).
Baseband OFDMA forward link BS transmitter.
OFDMA - Sub-carrier Allocation

- Clustered Carrier (CC-OFDMA): With CC-OFDMA, the $M$ sub-carriers are divided into $Q$ groups where each group consists of $N$ contiguous sub-carriers called clusters. The set of sub-carrier indices allocated to the $k$th user is $\{kN, kN + 1, \ldots, kN + N - 1\}$, where $0 \leq k < Q$. CC-OFDMA is sensitive to frequency-selective fading, because all sub-carriers assigned to a particular user may fade simultaneously.

- Spaced Carrier (SC-OFDMA): With SC-OFDMA, the $M$ sub-carriers are partitioned into $N$ groups, where each group has $Q$ contiguous sub-carriers. Then the $k$th sub-carrier of each group is assigned to the $k$th user. That is, the $k$th user is assigned the set of sub-carrier indices $\{k, Q+k, \ldots, (N-1)Q+k\}$, where $0 \leq k < Q$. SC-OFDMA is less sensitive to frequency-selective fading, since the sub-carriers assigned to each user span the entire bandwidth.

- Random Interleaving (RI-OFDMA): RI-OFDMA is used in IEEE802.16a. While the sub-carriers are partitioned into $N$ groups as in SC-OFDMA, the sub-carrier index in each of the $N$ groups that is assigned to a particular user is a random variable. The sub-carrier indices allocated to the $k$th user are $\{\epsilon_{k,1}, Q + \epsilon_{k,2}, \ldots, (M-1)Q + \epsilon_{k,M-1}\}$, where the $\epsilon_{k,i}$ are independent and identically distributed uniform random variables on the set $\{0, 1, \ldots, Q - 1\}$. 
OFDMA - Forward Link Receiver

- To remove the ISI introduced by the channel, the guard interval is removed. If the length of the cyclic prefix is at least as long as the discrete-time channel length, i.e., $G \geq L$, then we obtain the received sequence

$$R_{n,m} = R_{n,m}^g$$
$$= \sum_{i=0}^{L} g_i X_{n,(m-i)M} + \tilde{n}_{n,m}, \quad 0 \leq m \leq M - 1,$$

- Afterwards, an $M$-point IDFT is taken to transform to the frequency domain. This yields the output vector

$$z_{n,i} = \frac{1}{M} \sum_{m=0}^{M-1} R_{n,m} e^{-j \frac{2 \pi m i}{M}}$$
$$= T_i A x_{n,i} + \nu_{n,i}, \quad 0 \leq i \leq M - 1,$$

where

$$T_i = \sum_{m=0}^{L} g_m e^{-j \frac{2 \pi m i}{M}},$$

and the noise samples $\{\nu_{n,i}\}$ are i.i.d with zero-mean and variance $N_0/(MT_s^g)$.

- On the forward link each MS will only be interested in the $N$ data symbols that are transmitted by the BS on its allocated sub-carriers. Hence, only the DFT outputs with indices in the set $\mathcal{T}_j$ are used by the $j$th MS for data detection.
Baseband OFDMA forward link receiver.
**OFDMA - Reverse Link**

- On the OFDMA reverse link, $Q$ users transmit their signals to a central BS.
- Each MS transmitter only transmits its own data stream.
- Similar to the OFDMA forward link, the $j$th MS performs sub-carrier allocation, and the resulting vector $x_{j,n}$ is applied to an $M$-point IDFT, and appended with a length-$G$ cyclic guard interval.
- One of the biggest drawbacks of OFDMA is its high PAPR. A high PAPR may be acceptable on the forward link; however, a high PAPR is undesirable on the reverse link since the MS is often battery powered and amplifier back-off is required.

Baseband OFDMA reverse link MS transmitter.
SC-FDMA

- The SC-FDMA transmitter groups the modulation symbols into blocks of \( N \) symbols. Let

\[
x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,N})
\]

denote the \( n \)th block of modulation symbols. An \( N \)-point DFT (\( N \)-DFT) is taken on each block \( x_n \), to yield length-\( N \) vectors

\[
X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,N})
\]

that are the frequency domain representation of the blocks of input symbols.

- The sub-carrier mapper then maps the \( N \) components of the vector \( X_n \) onto a larger set of \( M \) sub-carriers such that \( M = NQ \), where \( Q \) is an integer. There are several different types of sub-carrier mappings, including the interleaved (I-FDMA) and localized (L-FDMA) mappings. The sub-carrier mapping generates the sequence \( \tilde{X}_n \).

- An \( M \)-point IDFT is then taken of the sequence \( \tilde{X}_n \) to produce the output sequence \( \tilde{x}_n \). The time domain input symbols \( x_{n,k} \) have duration \( T_s \) seconds. However, after going through the SC-FDMA modulator the time-domain output symbols \( \tilde{x}_{n,k} \) are compressed and have duration \( \tilde{T}_s = (N/M)T_s \) seconds.
**SC-FDMA Transmitter**

Baseband SC-FDMA transmitter. There are $M$ sub-carriers of which $N$ are occupied by the input data.
The SC-FDMA receiver first removes the cyclic guard interval.

Afterwards, an \( M \)-point DFT is taken to transform to the frequency domain.

Sub-carrier demapping and equalization is then performed in the frequency domain.

Finally, an \( N \)-point IDFT is used to convert the samples back to the time-domain for detection and further processing.

Baseband SC-FDMA receiver with SC-FDE.
SC-FDMA - Subcarrier Mapping

Interleaved FDMA (I-FDMA) subcarrier mapping.

Localized FDMA (L-FDMA) subcarrier mapping.