EE6604
Personal & Mobile Communications

Week 6

Mobile-to-Mobile and MIMO Channels

Fading Simulators
Haber & Akki Model

- Mobile-to-mobile (M-to-M) communication channels arise when both the transmitter and receiver are in motion and are equipped with low elevation antennas that are surrounded by local scatterers.

- Akki and Haber’s mathematical reference model for M-to-M flat fading channels gives the complex faded envelope as

\[
g(t) = \sqrt{\frac{1}{N}} \sum_{n=1}^{N} e^{j2\pi t \left( f_T T_m \cos(\alpha_T^{(n)}) + f_R R_m \cos(\alpha_R^{(n)}) \right)} + j\phi_n
\]

where

- \( N \) is the number of propagation paths
- \( f_T \) and \( f_R \) are the maximum Doppler frequencies due to the motion of the transmitter and receiver, respectively
- \( \alpha_T^{(n)} \) is the random angle of departure, and \( \alpha_R^{(n)} \) is the random angle of arrival, of the \( n^{th} \) propagation path measured with respect to the transmitter and receiver velocity vectors, respectively
- \( \phi_n \) is a random phase uniformly distributed on \( [-\pi, \pi) \) independent of \( \alpha_T^{(n)} \) and \( \alpha_R^{(n)} \) for all \( n \).
Time Correlation and Doppler Spectrum

• the ensemble averaged temporal correlation functions of the faded envelope are quite different and are as follows:

\[
\phi_{gIgI}(\tau) = \frac{1}{2}J_0(2\pi f_T^m \tau)J_0(2\pi a f_T^m \tau) \\
\phi_{gQgQ}(\tau) = \frac{1}{2}J_0(2\pi f_T^m \tau)J_0(2\pi a f_T^m \tau) \\
\phi_{gIgQ}(\tau) = \phi_{gQgI}(\tau) = 0 \\
\phi_{gg}(\tau) = \frac{1}{2}J_0(2\pi f_T^m \tau)J_0(2\pi a f_T^m \tau)
\]

where \( a = (f_R^m)/(f_T^m) \) is the ratio of the two maximum Doppler frequencies (or speeds) of the receiver and transmitter, and \( 0 \leq a \leq 1 \) assuming \( f_R^m \leq f_T^m \).

• Observe that the temporal correlation functions of M-to-M channels involve a product of two Bessel functions in contrast to the single Bessel function found in F-to-M channels.

• The corresponding Doppler spectrum, obtained by taking the Fourier transform of \( \phi_{gg}(\tau) \) is

\[
S_{gg}(f) = \frac{1}{\pi^2 f_T^m \sqrt{a}} K \left[ \frac{1 + a}{2\sqrt{a}} \sqrt{1 - \left( \frac{f}{(1 + a) f_T^m} \right)^2} \right]
\]

where \( K[\cdots] \) is the complete elliptic integral of the first kind.
Doppler spectrum for M-to-M and F-to-M channels.
Double Ring Model

To develop M-to-M channel simulation models, we apply a double-ring concept that defines two rings of isotropic scatterers placed, one around the transmitter and another around the receiver.

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Double ring model for M-to-M radio propagation with isotropic scattering at the transmitter and receiver.
Double Ring Model

- Using the double-ring model, the complex faded envelope can be written as

\[
g(t) = \sqrt{\frac{2}{NM}} \sum_{m=1}^{N} \sum_{n=1}^{M} \exp\left(-j2\pi \varepsilon_{mn}/\lambda_c\right) e^{j2\pi t}\left\{f_{m}^{T}\cos(\alpha_{T}^{(m)})+f_{R}^{R}\cos(\alpha_{R}^{(n)})\right\}+j\phi_{n,m},
\]

- The index ‘\(m\)’ refers to the paths traveling from the transmitter to the \(N\) scatterers located on the transmitter end ring, the index ‘\(n\)’ refers to the paths traveling from the \(M\) scatterers on the receiver end ring to the receiver.

- The angle \(\alpha_{T}^{(m)}\) is the random angle of departure and \(\alpha_{R}^{(n)}\) is the random angle of arrival of the \(\{m,n\}\)th propagation path measured with respect to the x-axis, respectively.

- The phases \(\phi_{n,m}\) are uniformly distributed on \([-\pi, \pi)\) and are independent for all pairs \(\{n,m\}\).

- Note that the single summation in the Haber & Akki model is replaced with a double summation, because each plane wave on its way from the transmitter to the receiver is double bounced.

  - The temporal correlation characteristics remain the same as those of the Haber & Akki model, because each path will undergo a Doppler shift due to the motion of both the transmitter and receiver.
MIMO Channels

- A multiple-input multiple-output (MIMO) system is one that consists of multiple transmit and receive antennas.
MIMO Channels

• For a system consisting of $L_t$ transmit and $L_r$ receive antennas, the channel can be described by $L_t \times L_r$ matrix

$$G(t, \tau) = \begin{bmatrix}
g_{1,1}(t, \tau) & g_{1,2}(t, \tau) & \cdots & g_{1,L_t}(t, \tau) 
g_{2,1}(t, \tau) & g_{2,2}(t, \tau) & \cdots & g_{2,L_t}(t, \tau) 
\vdots & \vdots & \ddots & \vdots 
g_{L_r,1}(t, \tau) & g_{L_r,2}(t, \tau) & \cdots & g_{L_r,L_t}(t, \tau)
g_{qp}(t, \tau) 
\end{bmatrix},$$

$g_{qp}(t, \tau)$ denotes the time-varying sub-channel impulse response between the $p$th transmitter antenna and $q$th receiver antenna.

• Suppose that the complex envelopes of the signals transmitted from the $L_t$ transmit antennas are:

$$\tilde{s}(t) = (\tilde{s}_1(t), \tilde{s}_2(t), \ldots, \tilde{s}_{L_t}(t))^T,$$

where $\tilde{s}_p(t)$ is the signal transmitted from the $p$th transmit antenna.

• Let

$$\tilde{r}(t) = (\tilde{r}_1(t), \tilde{r}_2(t), \ldots, \tilde{r}_{L_r}(t))^T,$$

denote the vector of received complex envelopes, where $\tilde{r}_q(t)$ is the signal received at the $q$th receiver antenna. Then

$$\tilde{r}(t) = \int_0^t G(t, \tau)\tilde{s}(t - \tau)d\tau$$
MIMO Channels - Special Cases

- Under conditions of flat fading

\[ G(t, \tau) = G(t)\delta(\tau - \hat{\tau}) , \]

where \( \hat{\tau} \) is the delay through the channel and

\[ \tilde{r}(t) = G(t)\tilde{s}(t - \hat{\tau}) . \]

- If the MIMO channel is characterized by slow fading, then

\[ \tilde{r}(t) = \int_0^t G(\tau)\tilde{s}(t - \tau)d\tau . \]

  - In this case, the channel matrix \( G(\tau) \) remains constant over the duration of the transmitted waveform \( \tilde{s}(t) \), but can vary from one channel use to the next, where a channel use may be defined as the transmission of either a single modulated symbol or a vector of modulated symbols.

  - Sometimes this is called a randomly static channel or a block fading channel.

- Finally, if the MIMO channel is characterized by slow flat fading, then

\[ \tilde{r}(t) = G\tilde{s}(t) . \]
MIMO Channel Models - Classification

- MIMO channel models can be classified as either physical or analytical models.
- The analytical models characterize the MIMO sub-channel impulse responses in a mathematical manner without explicitly considering the underlying electromagnetic wave propagation.
  - Analytical MIMO channel models are most often used under slowly and flat fading conditions.
  - The various analytical models generate the MIMO matrices as realizations of complex Gaussian random variables having specified means and correlations.
  - To model Rician fading, the channel matrix can be divided into a deterministic part and a random part, i.e.,

\[
G = \sqrt{\frac{K}{K+1}} \bar{G} + \sqrt{\frac{1}{K+1}} G_s
\]

where \(E[G] = \sqrt{\frac{K}{K+1}} \bar{G}\) is the LoS or specular component and \(\sqrt{\frac{K}{K+1}} G_s\) is the scatter component assumed to have zero-mean.
  - To simply our further characterization of the MIMO channel, assume for the time being that \(K = 0\), so that \(G = G_s\).
The simplest MIMO model assumes that the entries of the matrix \( G \) are independent and identically distributed (i.i.d) complex Gaussian random variables.

- This model corresponds to the so called ”rich scattering” or spatially white environment.
- Such an independence assumption simplifies the performance analysis of various digital signaling schemes operating on MIMO channels.
- In reality the sub-channels will be correlated and, therefore, the i.i.d. model will lead to optimistic performance estimates.
- A variety of more sophisticated models have been introduced to account for spatial correlation of the sub-channels.
Correlated MIMO Channel Models

• Consider the vector \( g = \text{vec}\{G\} \) where

\[
G = [g_1, g_2, \ldots, g_{Lt}] , \quad g_j = (g_{1,j}, g_{2,j}, \ldots, g_{Lr,j})^T
\]

and

\[
\text{vec}\{G\} = [g_1^T, g_2^T, \ldots, g_{Lt}^T]^T.
\]

• The vector \( g \) is a column vector of length \( n = LtL_r \). The vector \( g \) is zero-mean complex Gaussian random vector and its statistics are fully specified by the \( n \times n \) covariance matrix \( R_G = \frac{1}{2}E[gg^H] \), where \( g^H \) is the complex conjugate transpose of \( g \).

• Hence, \( g \sim \mathcal{CN}(0, R_G) \) and, if \( R_G \) is invertible, the probability density function of \( g \) is

\[
p(g) = \frac{1}{(2\pi)^{n\text{det}(R_G)}}e^{-\frac{1}{2}g^HR_G^{-1}g} , \quad g \in \mathbb{C}^n .
\]

• Realizations of the MIMO channel with the above distribution can be generated by

\[
G = \text{unvec}(g) \quad \text{with} \quad g = R_G^{1/2}w .
\]

Here, \( R_G^{1/2} \) is any matrix square root of \( R_G \), i.e., \( R_G = R_G^{1/2}(R_G^{1/2})^H \), and \( w \) is a length \( n \) vector where \( w \sim \mathcal{CN}(0, I) \).
Correlated MIMO Channel Models

- To find the square root of the matrix $R_G$, we can use eigenvalue decomposition.

- Note that the matrix $R_G$ is Hermitian, i.e., $R_G = R_G^H$.

- It follows that $R_G$ has the eigenvalue decomposition $R_G = U \Lambda U^H$, where $U$ is a unitary matrix, i.e., $UU^H = I$.

- Then we have $R_G^{1/2} = U \Lambda^{1/2} U^H$

- To verify this solution, we note that

$$R_G = U \Lambda^{1/2} U^H U \Lambda^{1/2} U^H$$

$$= U \Lambda^{1/2} \Lambda^{1/2} U^H$$

$$= U \Lambda U^H$$

- To find the matrix $U$, we formulate

$$R_G u = \lambda u$$

and solve for $\lambda$ and $u$. This can be done by solving the $N$ (assuming matrix $R_G$ is full rank) roots of the polynomial

$$p(\lambda) = \det(R_G - \lambda I) = 0$$

- For each solution $\lambda_i$, we have the specific eigenvalue equation which we solve for $u$

$$(R_G - \lambda I) u = 0$$
Kronecker Model

- The Kronecker model assumes that the spatial correlation at the transmitter and receiver is separable.
- This is equivalent to restricting the correlation matrix $R_H$ to have the Kronecker product form
  \[ R_G = R_T \otimes R_R \]
  where
  \[ R_T = \frac{1}{\sqrt{2}}E [G^H G] \quad R_R = \frac{1}{\sqrt{2}}E [GG^H] \]
  are the $L_t \times L_t$ and $L_r \times L_r$ transmit and receive correlation matrices respectively, and $\otimes$ is the “Kronecker product.”

  - For example, the Kronecker product of an $n \times n$ matrix $A$ and an $m \times m$ matrix $B$ would be
    \[ A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \]

- Under the above Kronecker assumption,
  \[ g = (R_T \otimes R_R)^{1/2} w \]
  and
  \[ G = R_R^{1/2} W R_T^{1/2} \]
  where $W$ is an $L_r \times L_t$ matrix consisting of i.i.d. zero mean complex Gaussian random variables.
Kronecker Model

• If the elements of $G$ could be arbitrarily selected, then the correlation functions would be a function of four sub-channel index parameters, i.e.,
\[ \frac{1}{2} \mathbb{E}[g_{qp}g_{\tilde{q}\tilde{p}}^*] = \phi(q, p, \tilde{q}, \tilde{p}) \]
where $g_{qp}$ is the channel between the $p$th transmit and $q$th receive antenna.

• However, due to the Kronecker property, $R_G = R_T \otimes R_R$, the elements of $G$ are structured.

• One implication of the Kronecker property is ”spatial” stationarity
\[ \frac{1}{2} \mathbb{E}[g_{qp}g_{\tilde{q}\tilde{p}}^*] = \phi(q - \tilde{q}, p - \tilde{p}) , \]
which implies that the sub-channel correlations are determined not by their position in the matrix $G$, but by their positional difference.

• In addition, to the stationary property, manipulation of the Kronecker product form in $R_G = R_T \otimes R_R$ implies that
\[ \frac{1}{2} \mathbb{E}[g_{qp}g_{\tilde{q}\tilde{p}}^*] = \phi(q - \tilde{q}, p - \tilde{p}) = \phi_R(q - \tilde{q}) \cdot \phi_T(p - \tilde{p}) , \]
meaning that the correlation can be separated into two parts: a transmitter part and a receiver part, and both parts are stationary.

• Finally, it can be shown that the Kronecker property, $R_G = R_T \otimes R_R$, holds if and only if the above separable property holds.
**Weichselberger Model**

- The Weichselberger model overcomes the separable requirement of the channel correlation functions of the Kronecker model.

- Consider the eigenvalue decomposition of the transmitter and receiver correlation matrices,

  \[
  R_T = U_T \Lambda_T U_T^H \\
  R_R = U_R \Lambda_R U_R^H
  \]

  \(\Lambda_T\) and \(\Lambda_R\) are diagonal matrices containing the eigenvalues of, and \(U_T\) and \(U_R\) are unity matrices containing the eigenvectors of, \(R_T\) and \(R_R\).

- The Weichselberger model constructs the matrix \(G\) as

  \[
  G = U_R (\hat{\Omega} \odot W) U_T^T
  \]

  where \(W\) is an \(L_r \times L_t\) matrix consisting of i.i.d. zero mean complex Gaussian random variables and \(\odot\) denotes the Schur-Hadamard product (element-wise matrix multiplication), and \(\Omega\) is an \(L_r \times L_t\) coupling matrix whose non-negative real values determine the average power coupling between the transmitter and receiver eigenvectors. The matrix \(\hat{\Omega}\) is the element-wise square root of \(\Omega\).

- The Kronecker model is a special case of the Weichselberger model obtained with the coupling matrix \(\Omega = \lambda_R \lambda_T^T\), where \(\lambda_R\) and \(\lambda_T\) are column vectors containing the eigenvalues of \(\Lambda_T\) and \(\Lambda_R\), respectively.
Filtered White Noise

- Since the complex faded envelope can be modelled as a complex Gaussian random process, one approach for generating the complex faded envelope is to filter a white noise process with appropriately chosen low pass filters

\[
\text{white Gaussian noise} \xrightarrow{\text{LPF } H(f)} g(t) \\
\text{white Gaussian noise} \xrightarrow{\text{LPF } H(f)} g'_I(t) + jg'_Q(t)
\]

- If the Gaussian noise sources are uncorrelated and have power spectral densities of \( \Omega_p/2 \) watts/Hz, and the low-pass filters have transfer function \( H(f) \), then

\[
S_{g_Ig_I}(f) = S_{g_Qg_Q}(f) = \frac{\Omega_p}{2} |H(f)|^2 \\
S_{g_Ig_Q}(f) = 0
\]

- Two approaches: IIR filtering method and IFFT filtering method
IIR Filtering Method

- implement the filters in the time domain as finite impulse response (FIR) or infinite impulse response (IIR) filters. There are two main challenges with this approach.

  - the normalized Doppler frequency, $\hat{f}_m = f_m T_s$, where $T_s$ is the simulation step size, is very small.
    * This can be overcome with an infinite impulse response (IIR) filter designed at a lower sampling frequency followed by an interpolator to increase the sampling frequency.

  - The second main challenge is that the square-root of the target Doppler spectrum for 2-D isotropic scattering and an isotropic antenna is irrational and, therefore, none of the straightforward filter design methods can be applied.
    * One possibility is to use the MATLAB function `iirlpnorm` to design the required filter.
IIR Filtering Method

- Here we consider an IIR filter of order \(2K\) that is synthesized as the the cascade of \(K\) Direct-Form II second-order (two poles and two zeroes) sections (biquads) having the form

\[
   H(z) = A \prod_{k=1}^{K} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}}.
\]

For example, for \(f_m T_s = 0.4\), \(K = 5\), and an ellipsoidal accuracy of 0.01, we obtain the coefficients tabulated below.

\begin{table}[h]
\centering
\begin{tabular}{c|cccc}
\hline
Stage & \(a_k\) & \(b_k\) & \(c_k\) & \(d_k\) \\
\hline
1 & 1.5806655278853 & 0.99720549234156 & -0.64808639835819 & 0.88900798545419 \\
2 & 0.19859624284546 & 0.99283177405702 & -0.62521063559242 & 0.97280125737779 \\
3 & -0.60387555371625 & 0.9999939585621 & -0.62031415619505 & 0.9999628706514 \\
4 & -0.56105447536557 & 0.9997677910713 & -0.7922029531477 & 0.2514924845181 \\
5 & -0.39828788982331 & 0.99957862369507 & -0.71405064745976 & 0.64701702807931 \\
A & 0.020939537466725 & & & \\
\hline
\end{tabular}
\end{table}

\textit{Coefficients for \(K = 5\) biquad stage elliptical filter, \(f_m T_s = 0.4\), \(K = 5\)}
Magnitude response of the designed shaping filter, $f_m T_s = 0.4$, $K = 5$. 
IFFT Filtering Method

IDFT-based fading simulator.
To implement 2-D isotropic scattering, the filter $H[k]$ can be specified as follows:

$$H[k] = \begin{cases} 
0 & , \ k = 0 \\
\sqrt{\frac{1}{2\pi f_m\sqrt{1-(k/(N f_m))^2}}} & , \ k = 1, 2, \ldots, k_m - 1 \\
\sqrt{k_m \left[ \frac{\pi}{2} - \arctan \left( \frac{k_m-1}{\sqrt{2k_m-1}} \right) \right]} & , \ k = k_m \\
0 & , \ k = k_m + 1, \ldots, N - k_m - 1 \\
\sqrt{k_m \left[ \frac{\pi}{2} - \arctan \left( \frac{k_m-1}{\sqrt{2k_m-1}} \right) \right]} & , \ k = N - k_m \\
\sqrt{\frac{1}{2\pi f_m\sqrt{1-(N-k/(N f_m))^2}}} & , \ N - k_m + 1, \ldots, N - 1
\end{cases}$$

One problem with the IFFT method is that the faded envelope is discontinuous from one block of $N$ samples to the next.
Sum of Sinusoids (SoS) Methods - Clarke’s Model

- With $N$ equal strength ($C_n = \sqrt{1/N}$) arriving plane waves

\[
g(t) = g_I(t) + jg_Q(t) = \sqrt{1/N} \sum_{n=1}^{N} \cos(2\pi f_m t \cos \theta_n + \phi_n) + j\sqrt{1/N} \sum_{n=1}^{N} \sin(2\pi f_m t \cos \theta_n + \phi_n) . \quad (1)
\]

- The normalization $C_n = \sqrt{1/N}$ makes $\Omega_p = 1$.

- The phases $\phi_n$ are independent and uniform on $[-\pi, \pi)$.

- With 2-D isotropic scattering, the $\theta_n$ are also independent and uniform on $[-\pi, \pi)$, and are independent of the $\phi_n$.

- Types of SoS simulators
  - deterministic - $\{\theta_n\}$ and $\{\phi_n\}$ are fixed for all simulation runs.
  - statistical - either $\{\theta_n\}$ or $\{\phi_n\}$, or both, are random for each simulation run.
  - ergodic statistical - either $\{\theta_n\}$ or $\{\phi_n\}$, or both, are random, but only a single simulation run is required.
Clarke’s Model - Ensemble Averages

• The statistical properties of Clarke’s model in for finite $N$ are

\[
\phi_{g_1g_1}(\tau) = \phi_{g_Qg_Q}(\tau) = \frac{1}{2} J_0(2\pi f_m \tau)
\]

\[
\phi_{g_1g_Q}(\tau) = \phi_{g_Qg_1}(\tau) = 0
\]

\[
\phi_{gg}(\tau) = \frac{1}{2} J_0(2\pi f_m \tau)
\]

\[
\phi_{|g|^2|g|^2}(\tau) = \mathbb{E}[|g|^2(t)|g|^2(t + \tau)]
\]

\[
= 1 + \frac{N - 1}{N} J_0^2(2\pi f_m \tau)
\]

• For finite $N$, the ensemble averaged auto- and cross-correlation of the quadrature components match those of the 2-D isotropic scattering reference model.

• The squared envelope autocorrelation reaches the desired form $1 + J_0^2(2\pi f_m \tau)$ asymptotically as $N \to \infty$. 
Clarke’s Model - Time Averages

- In simulations, time averaging is often used in place of ensemble averaging. The corresponding time average correlation functions \( \hat{\phi}(\cdot) \) (all time averaged quantities are distinguished from the statistical averages with a ‘ \( \hat{\cdot} \)’) are random and depend on the specific realization of the random parameters in a given simulation trial.

- The variances of the time average correlation functions, defined as

\[
\text{Var}[\hat{\phi}(\cdot)] = E \left[ \left| \hat{\phi}(\cdot) - \lim_{N \to \infty} \phi(\cdot) \right|^2 \right],
\]

characterizes the closeness of a simulation trial with finite \( N \) and the ideal case with \( N \to \infty \).

- These variances can be derived as follows:

\[
\begin{align*}
\text{Var}[\hat{\phi}_{gIgI}(\tau)] &= \text{Var}[\hat{\phi}_{gQgQ}(\tau)] \\
&= \frac{1 + J_0(4\pi f_m \tau) - 2J_0^2(2\pi f_m \tau)}{8N} \\
\text{Var}[\hat{\phi}_{gIgQ}(\tau)] &= \text{Var}[\hat{\phi}_{QgI}(\tau)] \\
&= \frac{1 - J_0(4\pi f_m \tau)}{8N} \\
\text{Var}[\hat{\phi}_{gg}(\tau)] &= \frac{1 - J_0^2(2\pi f_m \tau)}{4N}
\end{align*}
\]
Jakes’ Deterministic Method

• To approximate an isotropic scattering channel, it is assumed that the $N$ arriving plane waves uniformly distributed in angle of incidence:

$$\theta_n = 2\pi n / N \ , \ n = 1, 2, \ldots, N$$

• By choosing $N/2$ to be an odd integer, the sum in (1) can be rearranged into the form

$$g(t) = \sqrt{\frac{1}{N}} \sum_{n=1}^{N/2-1} \left[ e^{-j(2\pi f_m t \cos \theta_n + \hat{\phi}_n)} + e^{j(2\pi f_m t \cos \theta_n + \hat{\phi}_n)} \right]$$

$$+ e^{-j(2\pi f_m t + \hat{\phi}_N)} + e^{j(2\pi f_m t + \hat{\phi}_N)}$$

(2)

• The Doppler shifts progress from $-2\pi f_m \cos(2\pi / N)$ to $+2\pi f_m \cos(2\pi / N)$ as $n$ progresses from 1 to $N/2 - 1$ in the first sum, while in the second sum they progress from $+2\pi f_m \cos(2\pi / N)$ to $-2\pi f_m \cos(2\pi / N)$.

• Jakes uses nonoverlapping frequencies to write $g(t)$ as

$$g(t) = \sqrt{2} \sqrt{\frac{1}{N}} \sum_{n=1}^{M} \left[ e^{-j(\hat{\phi}_n + 2\pi f_m t \cos \theta_n)} + e^{j(\hat{\phi}_n + 2\pi f_m t \cos \theta_n)} \right]$$

$$+ e^{-j(\hat{\phi}_N + 2\pi f_m t)} + e^{j(\hat{\phi}_N + 2\pi f_m t)}$$

(3)

where

$$M = \frac{1}{2} \left( \frac{N}{2} - 1 \right)$$

and the factor $\sqrt{2}$ is included so that the total power remains unchanged.
• Note that (2) and (3) are not equal. In (2) all phases are independent. However, (3) implies that $-\hat{\phi}_i = \hat{\phi}_{N/2-i}$ and, therefore, correlation is introduced into the phases.

• Jakes’ further imposes the constraint $\hat{\phi}_n = -\hat{\phi}_{-n}$ to give

$$g(t) = \sqrt{2} \left\{ 2 \sum_{n=1}^{M} \cos \beta_n \cos 2\pi f_n t + \sqrt{2} \cos \alpha \cos 2\pi f_m t \right\}$$

$$+ j \left\{ 2 \sum_{n=1}^{M} \sin \beta_n \cos 2\pi f_n t + \sqrt{2} \sin \alpha \cos 2\pi f_m t \right\}$$

where

$$\alpha = \hat{\phi}_N = -\hat{\phi}_{-N} \quad \beta_n = \hat{\phi}_n = -\hat{\phi}_{-n} \quad M = \frac{1}{2} \left( \frac{N}{2} - 1 \right)$$
Jakes’ fading simulator that generates a faded envelope by summing waveforms from $M + 1$ low frequency oscillators.
• Time averages:

\[
\langle g_I^2(t) \rangle = 2 \sum_{n=1}^{M} \cos^2 \beta_n + \cos^2 \alpha \\
= M + \cos^2 \alpha + \sum_{n=1}^{M} \cos 2\beta_n
\]

\[
\langle g_Q^2(t) \rangle = 2 \sum_{n=1}^{M} \sin^2 \beta_n + \sin^2 \alpha \\
= M + \sin^2 \alpha - \sum_{n=1}^{M} \cos 2\beta_n
\]

\[
\langle g_I(t)g_Q(t) \rangle = 2 \sum_{n=1}^{M} \sin \beta_n \cos \beta_n + \sin \alpha \cos \alpha.
\]

• Choose the \( \beta_n \) and \( \alpha \) so that \( g_I(t) \) and \( g_Q(t) \) have zero-mean, equal variance, and zero cross-correlation.

• The choices \( \alpha = 0 \) and \( \beta_n = \pi n / M \) will yield \( \langle g_Q^2(t) \rangle = M, \langle g_I^2(t) \rangle = M + 1, \) and \( \langle g_I(t)g_Q(t) \rangle = 0. \)

• The envelope power \( \langle g_I^2(t) \rangle + \langle g_Q^2(t) \rangle \) can be scaled to any desired value.
Typical faded envelope generated with 8 oscillators.
Auto- and Cross-correlations

- The normalized autocorrelation function is
  \[ \phi_{gg}^n(\tau) = \frac{E[g^*(t)g(t + \tau)]}{E[|g(t)|^2]} \]

- With 2-D isotropic scattering
  \[ \phi_{g_Ig_I}(\tau) = \phi_{g_Qg_Q}(\tau) = \frac{\Omega_p}{2} J_0(2\pi f_m \tau) \]
  \[ \phi_{g_Ig_Q}(\tau) = \phi_{g_Qg_I}(\tau) = 0 \]

- Therefore,
  \[ \phi_{gg}^n(\tau) = \frac{E[g^*(t)g(t + \tau)]}{E[|g(t)|^2]} = J_0(2\pi f_m \tau) \]
Auto- and Cross-correlations

- For Clarke’s model with angles $\theta_n$ that are independent and uniform on $[-\pi, \pi)$, the normalized autocorrelation function is

\[ \phi^g_{gg}(\tau) = \frac{E[g^*(t)g(t+\tau)]}{E[|g(t)|^2]} = J_0(2\pi f_m \tau). \]

- Clark’s model with even $N$ and the restriction $\theta_n = \frac{2\pi n}{N}$, yields the normalized ensemble averaged autocorrelation function

\[ \phi^n_{gg}(\tau) = \frac{1}{N} \sum_{n=1}^{N} \cos \left( 2\pi f_m \tau \cos \frac{2\pi n}{N} \right). \]

  - Clark’s model with $\theta_n = \frac{2\pi n}{N}$ yields an autocorrelation function that deviates from the desired values at large lags.

- Finally, the normalized time averaged autocorrelation function for Jakes’ method is

\[ \phi^n_{gg}(t, t+\tau) = \frac{2}{N} \left( \cos 2\pi f_m \tau + \cos 2\pi f_m (2t + \tau) \right) \]

\[ + \frac{4}{N} \sum_{n=1}^{M} \left( \cos 2\pi f_n \tau + \cos 2\pi f_n (2t + \tau) \right) \]

  - Jakes’ simulator is not wide-sense stationary.
Autocorrelation of inphase and quadrature components obtained with Clarke’s method, using $\theta_n = \frac{2\pi n}{N}$ and $N = 8$ oscillators.
Autocorrelation of inphase and quadrature components obtained with Clarke’s method, using $\theta_n = \frac{2\pi n}{N}$ and $N = 16$ oscillators.