EE6604
Personal & Mobile Communications

Week 12

Power Spectrum of Digitally Modulated Signals
POWER SPECTRAL DENSITIES

• A bandpass modulated signal can be written in the form

\[ s(t) = \Re \{ \tilde{s}(t)e^{j(2\pi f_c t)} \} \]
\[ = \frac{1}{2} \{ \tilde{s}(t)e^{j(2\pi f_c t)} + \tilde{s}^*(t)e^{-j(2\pi f_c t)} \} \]

• The autocorrelation of a bandpass modulated signal is

\[ \phi_{ss}(\tau) = E[s(t)s(t + \tau)] \]
\[ = \frac{1}{4} E \left[ (\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}^*(t)e^{-j2\pi f_c t}) \right. \]
\[ \times \left( \tilde{s}(t + \tau)e^{j(2\pi f_c t + 2\pi f_c \tau)} + \tilde{s}^*(t + \tau)e^{-j(2\pi f_c t + 2\pi f_c \tau)} \right) \]
\[ = \frac{1}{4} E \left[ \tilde{s}(t)\tilde{s}(t + \tau)e^{j(4\pi f_c t + 2\pi f_c \tau)} + \tilde{s}(t)\tilde{s}^*(t + \tau)e^{-j(2\pi f_c t + 2\pi f_c \tau)} \right. \]
\[ + \tilde{s}^*(t)\tilde{s}(t + \tau)e^{j2\pi f_c \tau} + \tilde{s}^*(t)\tilde{s}^*(t + \tau)e^{-j(4\pi f_c t + 2\pi f_c \tau)} \]
\[ \left. + E[\tilde{s}^*(t)\tilde{s}(t + \tau)]e^{j2\pi f_c \tau} + E[\tilde{s}^*(t)\tilde{s}^*(t + \tau)]e^{-j(2\pi f_c t + 2\pi f_c \tau)} \right] \].
• If $s(t)$ is a wide-sense stationary random process, then the exponential terms that involve $t$ must vanish, i.e., $E[\tilde{s}(t)\tilde{s}(t + \tau)] = 0$ and $E[\tilde{s}^*(t)\tilde{s}^*(t + \tau)] = 0$.

• Substituting $\tilde{s}(t) = \tilde{s}_I(t) + j\tilde{s}_Q(t)$ into the above expectations gives the result

\[
\phi_{\tilde{s}_I\tilde{s}_I}(\tau) = E[\tilde{s}_I(t)\tilde{s}_I(t + \tau)] = E[\tilde{s}_Q(t)\tilde{s}_Q(t + \tau)] = \phi_{\tilde{s}_Q\tilde{s}_Q}(\tau)
\]
\[
\phi_{\tilde{s}_I\tilde{s}_Q}(\tau) = E[\tilde{s}_I(t)\tilde{s}_Q(t + \tau)] = -E[\tilde{s}_Q(t)\tilde{s}_I(t + \tau)] = -\phi_{\tilde{s}_Q\tilde{s}_I}(\tau)
\]

• Using these results, the autocorrelation is

\[
\phi_{ss}(\tau) = \frac{1}{2}\phi_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau} + \frac{1}{2}\phi_{\tilde{s}\tilde{s}}^*(\tau)e^{-j2\pi f_c\tau}
\]

where

\[
\phi_{\tilde{s}\tilde{s}}(\tau) = \frac{1}{2}E[\tilde{s}^*(t)\tilde{s}(t + \tau)]
\]

• The power density spectrum is the Fourier transform of $\phi_{ss}(\tau)$:

\[
S_{ss}(f) = \frac{1}{2}[S_{\tilde{s}\tilde{s}}(f - f_c) + S_{\tilde{s}\tilde{s}}^*(-f - f_c)]
\]

$S_{\tilde{s}\tilde{s}}(f)$ is the power density spectrum of the complex envelope $\tilde{s}(t)$, which is always real-valued but not necessarily even about $f = 0$.

\[
S_{ss}(f) = \frac{1}{2}[S_{\tilde{s}\tilde{s}}(f - f_c) + S_{\tilde{s}\tilde{s}}(-f - f_c)]
\]
POWER SPECTRAL DENSITY OF A COMPLEX ENVELOPE

- In general, the complex lowpass signal is of the form
  \[ \tilde{s}(t) = A \sum_k b(t - kT, x_k) \]

- The autocorrelation of \( \tilde{s}(t) \) is
  \[
  \phi_{\tilde{s}\tilde{s}}(t, t + \tau) = \frac{1}{2} E [\tilde{s}^*(t)\tilde{s}(t + \tau)]
  = \frac{A^2}{2} \sum_i \sum_k E [b^*(t - iT, x_i) b(t + \tau - kT, x_k)] .
  \]

Observe that \( \tilde{s}(t) \) is a cyclostationary random process, meaning that the autocorrelation function \( \phi_{\tilde{s}\tilde{s}}(t, t + \tau) \) is periodic in \( t \) with period \( T \). To see this property, first note that

\[
\phi_{\tilde{s}\tilde{s}}(t + T, t + T + \tau)
= \frac{A^2}{2} \sum_i \sum_k E [b^*(t + T - iT, x_i) b(t + T + \tau - kT, x_k)]
= \frac{A^2}{2} \sum_{i'} \sum_{k'} E [b^*(t - i'T, x_{i'+1}) b(t + \tau - k'T, x_{k'+1})] .
\]
• Under the assumption that the information sequence is a stationary random process we can write

$$\phi_{\tilde{s}\tilde{s}}(t + T, t + T + \tau) = \frac{A^2}{2} \sum_{i'} \sum_{k'} \mathbb{E} \left[ b^*(t - i'T, x_{i'}) b(t + \tau - k'T, x_{k'}) \right]$$

$$= \phi_{\tilde{s}\tilde{s}}(t, t + \tau) . \tag{1}$$

where data blocks $x_{i'+1}$ and $x_{k'+1}$ are replaced by $x_{i'}$ and $x_{k'}$, respectively. Therefore $\tilde{s}(t)$ is cyclostationary.
Since $\tilde{s}(t)$ is cyclostationary, the autocorrelation $\phi_{\tilde{s}\tilde{s}}(\tau)$ can be obtained by taking the time average of $\phi_{\tilde{s}\tilde{s}}(t, t + \tau)$, given by

$$
\phi_{\tilde{s}\tilde{s}}(\tau) = < \phi_{\tilde{s}\tilde{s}}(t, t + \tau) >
$$

$$
= \frac{A^2}{2} \sum_i \sum_k \frac{1}{T} \int_0^T E \left[ b^*(t - iT, x_i) b(t + \tau - kT, x_k) \right] dt
$$

$$
= \frac{A^2}{2T} \sum_i \sum_k \int_{-iT}^{iT} E \left[ b^*(z, x_i) b(z + \tau - (k - i)T, x_k) \right] dz
$$

$$
= \frac{A^2}{2T} \sum_i \sum_m \int_{-iT}^{iT} E \left[ b^*(z, x_i) b(z + \tau - mT, x_{m+i}) \right] dz
$$

$$
= \frac{A^2}{2T} \sum_i \sum_m \int_{-iT}^{iT} E \left[ b^*(z, x_0) b(z + \tau - mT, x_m) \right] dz
$$

$$
= \frac{A^2}{2T} \sum_m \int_{-\infty}^{\infty} E \left[ b^*(z, x_0) b(z + \tau - mT, x_m) \right] dz
$$

where $< \cdot >$ denotes time averaging and the second last equality used the stationary property of the data sequence $\{x_k\}$. 
The psd of $\tilde{s}(t)$ is obtained by taking the Fourier transform of $\phi_{\tilde{s}\tilde{s}}(\tau)$,

$$S_{\tilde{s}\tilde{s}}(f) = \mathbb{E} \left[ \frac{A^2}{2T} \sum_m \int_{-\infty}^{\infty} b^*(z, x_0) b(z + \tau - mT, x_m) dz e^{-j2\pi f T d\tau} \right]$$

$$= \mathbb{E} \left[ \frac{A^2}{2T} \sum_m \int_{-\infty}^{\infty} b^*(z, x_0) e^{j2\pi f z} dz \times \int_{-\infty}^{\infty} b(z + \tau - mT, x_m) e^{-j2\pi f (z+\tau-mT)} d\tau e^{-j2\pi fmT} \right]$$

$$= \mathbb{E} \left[ \frac{A^2}{2T} \sum_m \int_{-\infty}^{\infty} b^*(z, x_0) e^{-j2\pi f z} d\tau \int_{-\infty}^{\infty} b(\tau', x_m) e^{-j2\pi f \tau'} d\tau' e^{-j2\pi fmT} \right]$$

$$= \frac{A^2}{2T} \sum_m \mathbb{E} \left[ B^*(f, x_0) B(f, x_m) \right] e^{-j2\pi fmT},$$

where $B(f, x_m)$ is the Fourier transform of $b(t, x_m)$.

Finally,

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} \sum_m S_{b,m}(f) e^{-j2\pi fmT}$$

where

$$S_{b,m}(f) = \frac{1}{2} \mathbb{E} \left[ B^*(f, x_0) B(f, x_m) \right]$$
• Suppose that $x_m$ and $x_0$ are uncorrelated for $|m| \geq K$.

Then

$$S_{b,m}(f) = S_{b,K}(f), \quad |m| \geq K$$

where

$$S_{b,K}(f) = \frac{1}{2} E[B^*(f, x_0)] E[B(0, x_m)] \quad |m| \geq K$$

$$= \frac{1}{2} E[B^*(f, x_0)] E[B(f, x_0)] \quad |m| \geq K$$

$$= \frac{1}{2} |E[B(f, x_0)]|^2, \quad |m| \geq K.$$

• It follows that

$$S_{\tilde{s}_s}(f) = S_{\tilde{s}_s}^c(f) + S_{\tilde{s}_s}^d(f)$$

where

$$S_{\tilde{s}_s}^c(f) = \frac{A^2}{T} \sum_{|m| < K} (S_{b,m}(f) - S_{b,K}(f)) e^{-j2\pi fmT}$$

$$S_{\tilde{s}_s}^d(f) = \left(\frac{A}{T}\right)^2 S_{b,K}(f) \sum_n \delta(f - \frac{n}{T})$$

• Note that the spectrum consists of discrete and continuous parts. The discrete portion has spectral lines spaced at $1/T$ Hz apart.
ZERO MEAN SIGNALS

- If $\tilde{s}(t)$ has zero mean, i.e., $E[b(t, x_0)] = 0$, then $E[B(f, x_0)] = 0$.
- Under this condition
  \[ S_{b,K}(f) = \frac{1}{2} |E[B(f, x_0)]|^2 = 0 \]
- Hence, $S_{\tilde{s}\tilde{s}}(f)$ has no discrete component and
  \[ S_{\tilde{s}\tilde{s}}(f) = \left( \frac{A^2}{T} \right) \sum_{|m| < K} S_{b,m}(f)e^{-j2\pi fmT} \]
UNCORRELATED SOURCE SYMBOLS

• With uncorrelated source symbols the information symbols \( x_{m,k} \) constituting data blocks \( \mathbf{x}_m = (x_{m,1}, x_{m,2}, \ldots, x_{m,N}) \) are mutually uncorrelated. Under this condition \( \mathbf{x}_m \) and \( \mathbf{x}_0 \) are obviously uncorrelated for \( |m| \geq 1 \).

• Hence, \( S_{b,m}(f) = S_{b,1}(f) \), for \( |m| \geq 1 \), where

\[
S_{b,0}(f) = \frac{1}{2} \mathbb{E}[|B(f, \mathbf{x}_0)|^2]
\]

\[
S_{b,1}(f) = \frac{1}{2} |\mathbb{E}[B(f, \mathbf{x}_0)]|^2
\]

• Hence

\[
S^d_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T^2} S_{b,1}(f) \sum_n \delta(f - n/T)
\]

\[
S^c_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} (S_{b,0}(f) - S_{b,1}(f))
\]

• If \( \tilde{s}(t) \) has zero mean as well, then

\[
S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} S_{b,0}(f)
\]
LINEAR FULL RESPONSE MODULATION

• Here we assume that

\[ b(t, x_k) = x_k h_a(t) \]
\[ B(f, x_k) = x_k H_a(f) \]

where the \( x_k \) may be correlated.

• Using the above we have the following

\[
S_{b,m}(f) = \frac{1}{2} \mathbb{E} \left[ B^*(f, x_0) B(f, x_m) \right] \\
= \frac{1}{2} \mathbb{E} \left[ x_0^* H_a^*(f) x_m H_a(f) \right] \\
= \frac{1}{2} \mathbb{E} \left[ x_0^* x_m |H_a(f)|^2 \right] \\
= \frac{1}{2} \mathbb{E} \left[ x_0^* x_m |H_a(f)|^2 \right] \\
= \phi_{xx}(m) |H_a(f)|^2
\]

where

\[
\phi_{xx}(m) = \frac{1}{2} \mathbb{E}[x_k^* x_{k+m}]
\]
• The psd of the complex envelope is

\[ S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} \sum_{m} S_{b,m}(f)e^{-j2\pi fmT} \]

\[ = \frac{A^2}{T} |H_a(f)|^2 \sum_{m} \phi_{xx}(m)e^{-j2\pi fmT} \]

\[ = \frac{A^2}{T} |H_a(f)|^2 S_{xx}(f) \]

where

\[ S_{xx}(f) = \sum_{m} \phi_{xx}(m)e^{-j2\pi fmT} \]

• With uncorrelated source symbols

\[ S_{b,0}(f) = \sigma_x^2 |H_a(f)|^2 \]

\[ S_{b,m}(f) = \frac{1}{2} |\mu_x|^2 |H_a(f)|^2 , \quad |m| \geq 1 . \]

where \( \sigma_x^2 = \frac{1}{2} |x_k|^2 \), \( \mu_x = \text{E}[x_k] \).

• If \( \mu_x = 0 \), then \( S_{b,1}(f) = 0 \) and

\[ S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} \sigma_x^2 |H_a(f)|^2 \]
POWER SPECTRAL DENSITY OF ASK

Psd of ASK with a truncated square root raised cosine pulse with various truncation lengths; $\beta = 0.5$. 
OFDM Power Spectrum

• The data symbols $x_{n,k}, k = 0, \ldots, N - 1$ that modulate the $N$ sub-carriers are assumed to have zero mean, variance $\sigma_x^2 = \frac{1}{2} \mathbb{E}[|x_{n,k}|^2]$, and they are mutually uncorrelated.

• In this case, the psd of the OFDM waveform is

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T_g} S_{b,0}(f),$$

where

$$S_{b,0}(f) = \frac{1}{2} \mathbb{E}[|B(f, x_0)|^2],$$

and

$$B(f, x_0) = \sum_{k=0}^{N-1} x_{0,k} T \text{sinc}(fT - k) + \sum_{k=0}^{N-1} x_{0,k} \alpha_g T \text{sinc}(\alpha_g(fT - k)) e^{j2\pi fT}.$$ 

Using the above along with $T = NT_s$ yields the result

$$S_{\tilde{s}\tilde{s}}(f) = \sigma_x^2 A^2 T \left( \frac{1}{1 + \alpha_g} \sum_{k=0}^{N-1} \text{sinc}^2(NfT_s - k) \right)$$

$$+ \frac{\alpha_g^2}{1 + \alpha_g} \sum_{k=0}^{N-1} \text{sinc}^2(\alpha_g(NfT_s - k))$$

$$+ \frac{2\alpha_g}{1 + \alpha_g} \cos(2\pi NfT_s) \sum_{k=0}^{N-1} \text{sinc}(NfT_s - k) \text{sinc}(\alpha_g(NfT_s - k)) \right).$$

• Note that the Nyquist frequency in this case is $1/2T_s^g = (1 + \alpha_g)/2T_s$. 
Psd of OFDM with $N = 16$, $\alpha_g = 0$. 
The PSD of OFDM with $N = 16$, $\alpha_g = 0.25$. 

The graph shows the power spectral density $S_{ss}(f)$ in dB against $fT_s$.
Psd of OFDM with $N = 1024, \alpha_g = 0$. 
Psd of OFDM with $N = 1024, \alpha_g = 0.25$. 
The output of the IDFT baseband modulator is \( \{ X^g \} = \{ X^g_{n,m} \} \), where \( m \) is the block index and

\[
X^g_{n,m} = X_{n,(m)N} = A \sum_{k=0}^{N-1} x_{n,k} e^{j2\pi km/N}, \quad m = 0, 1, \ldots, N + G - 1
\]

The power spectrum of the sequence \( \{ X^g \} \) can be calculated by first determining the discrete-time autocorrelation function of the time-domain sequence \( \{ X^g \} \) and then taking a discrete-time Fourier transform of the discrete-time autocorrelation function.

The psd of the OFDM complex envelope with ideal DACs can be obtained by applying the resulting power spectrum to an ideal low-pass filter with a cutoff frequency of \( 1/(2T^g_s) \) Hz.
Discrete-time Autocorrelation Function

- The time-domain sequence \( \{X^g\} \) is a periodic wide-sense stationary sequence having the discrete-time autocorrelation function

\[
\phi_{X^gX^g}(m, \ell) = \frac{1}{2} \mathbb{E}[X_{n,m}^g(X_{n,m+\ell}^g)^*] = A^2 \sum_{k=0}^{N-1} \sum_{i=0}^{N-1} \frac{1}{2} \mathbb{E}[x_{n,k}x_{n,i}^*]e^{j2\pi (km-im-\ell)/N},
\]

for \( m = 0, \ldots, N + G - 1 \).

The data symbols, \( x_{n,k} \), are assumed to be mutually uncorrelated with zero mean and variance \( \sigma_x^2 = \frac{1}{2} \mathbb{E}[|x_{n,k}|^2] \). Using the fact, that \( X_{n,m}^g = X_{n,(m)_N} \), we have the following result

\[
\phi_{X^gX^g}(m, \ell) = \begin{cases} 
A\sigma_x^2 & m = 0, \ldots, G - 1, \ell = 0, N \\
A\sigma_x^2 & m = G, \ldots, N - 1, \ell = 0 \\
A\sigma_x^2 & m = N, \ldots, N + G - 1, \ell = 0, -N \\
0 & \text{otherwise}
\end{cases}.
\]

Averaging over all time indices \( m \) in a block gives the time-average discrete-time autocorrelation function

\[
\phi_{X^gX^g}(\ell) = \begin{cases} 
A\sigma_x^2 & \ell = 0 \\
\frac{G}{N+G}A\sigma_x^2 & \ell = -N, N \\
0 & \text{otherwise}
\end{cases}.
\]
Taking the discrete-time Fourier transform of the discrete-time autocorrelation function gives

\[ S_{XgXg}(f) = \sum_m \phi_{XgXg}(\ell) e^{-j2\pi f m N T_s^g} \]

\[ = A\sigma_x^2 \left( 1 + \frac{G}{N+G} e^{-j2\pi f N T_s^g} + \frac{G}{N+G} e^{j2\pi f N T_s^g} \right) \]

\[ = A\sigma_x^2 \left( 1 + \frac{2G}{N+G} \cos(2\pi f N T_s^g) \right) . \]

Finally, we assume that the sequence \( \{X^g\} = \{X^g_{n,m}\} \) is passed through an ideal DACs.

– The ideal DAC is a low-pass filter with cutoff frequency \( 1/(2T_s^g) \).

The OFDM complex envelope has the psd

\[ S_{ss}(f) = A\sigma_x^2 \left( 1 + \frac{2G}{N+G} \cos(2\pi f N T_s^g) \right) \text{rect}(f T_s^g) . \]
Psd of IDFT-based OFDM with $N = 16, G = 0$. Note in this case that $T_s^g = T_s$. 
Psd of IDFT-based OFDM with $N = 16, G = 4$. 
Psd of IDFT-based OFDM with $N = 1024$, $G = 256$. 
Power spectral density of binary CPFSK for various modulation indices.
Psd of binary CPFSK as the modulation index $h \rightarrow 1$. 
Power spectral density of GMSK with various normalized filter bandwidths $BT$. 