Week 3

Flat Fading Channels
Envelope Distribution
A typical macrocellular mobile radio environment.
Multipath-Fading Mechanism

Typical mobile-to-mobile radio propagation environment.
Path loss, shadowing, envelope fading.
2-D Model of a typical wave component incident on a mobile station (MS).

- Assuming 2-D propagation, the Doppler shift is \( f_{D,n} = f_m \cos \theta_n \), where \( f_m = v/\lambda_c \) (\( \lambda_c \) is the carrier wavelength, \( v \) is the mobile station velocity).
Consider the transmission of the band-pass signal

\[ s(t) = \text{Re}\{\tilde{s}(t)e^{j2\pi f_c t}\} \]

At the receiver antenna, the \( n \)th plane wave arrives at angle \( \theta_n \) and experiences Doppler shift \( f_{D,n} = f_m \cos \theta_n \) and propagation delay \( \tau_n \).

If there are \( N \) propagation paths, the received bandpass signal is

\[ r(t) = \text{Re}\left[ \sum_{n=1}^{N} C_n e^{j\phi_n - j\frac{2\pi c \tau_n}{\lambda_c} + j2\pi (f_c + f_{D,n})t} \tilde{s}(t - \tau_n) \right] , \]

where \( C_n, \phi_n, f_{D,n} \) and \( \tau_n \) are the amplitude, phase, Doppler shift and time delay, respectively, associated with the \( n \)th propagation path, and \( c \) is the speed of light.

The delay \( \tau_n = d_n/c \) is the propagation delay associated with the \( n \)th propagation path, where \( d_n \) is the length of the path. The path lengths, \( d_n \), will depend on the physical scattering geometry which we have not specified at this point.
Multipath Propagation

- The received bandpass signal $r(t)$ has the form

$$r(t) = \text{Re} \left[ \tilde{r}(t) e^{j2\pi f_c t} \right]$$

where the received complex envelope is

$$\tilde{r}(t) = \sum_{n=1}^{N} C_n e^{j\phi_n(t)} \tilde{s}(t - \tau_n)$$

and

$$\phi_n(t) = \phi_n - 2\pi c\tau_n / \lambda_c + 2\pi f_{D,n} t$$

is the time-variant phase associated with the $n$th propagation path.

- Note that the $n$th component varies with the Doppler frequency $f_{D,n}$.
- The term $c\tau_n / \lambda_c$ is the propagation distance $c\tau_n$ normalized by the carrier wavelength $\lambda_c$. For cellular frequencies (900 MHz), $\lambda_c$ is on the order of a foot.
- The phase $\phi_n$ is a random introduced by the $n$th scatterer and can be assumed to be uniformly distributed on $[-\pi, \pi)$.
- The received phase at any time $t$, $\phi_n(t)$ is uniformly distributed on $[-\pi, \pi)$. 
Flat Fading - impulse response

- The received waveform is given by the convolution

\[ \tilde{r}(t) = \int_0^t g(t, \tau) \tilde{s}(t - \tau) d\tau \]

- It follows that the channel can be modeled by a linear time-variant filter having the time-variant impulse response

\[ g(t, \tau) = \sum_{n=1}^{N} C_n e^{j\phi_n(t)} \delta(\tau - \tau_n) \]

- If the differential path delays \( \tau_i - \tau_j \) are all very small compared to the modulation symbol period, \( T \), then the \( \tau_n \) can be replaced by the mean delay \( \mu_\tau \) inside the delta function. Note that this approximation is not applied to the channel phases \( \phi_n(t) \), since small changes in \( \tau_n \) result in large changes in \( \phi_n(t) \).

  - The channel impulse response has the approximate form

\[ g(t, \tau) = g(t) \delta(\tau - \mu_\tau) , \quad g(t) = \sum_{n=1}^{N} C_n e^{j\phi_n(t)} . \]

  - The received complex envelope is

\[ \tilde{r}(t) = g(t) \tilde{s}(t - \mu_\tau) \] (1)

which experiences **fading** due to the time-varying complex channel gain \( g(t) \).
Flat Fading - frequency domain

- By taking Fourier transforms of both sides of (1), the received complex envelope in the frequency domain is

\[ \tilde{R}(f) = G(f) \ast \tilde{S}(f)e^{-j2\pi f \mu_r} \]

- Since the channel component \( g(t) \) changes with time, it follows that \( G(f) \) has a finite non-zero width in the frequency domain.

- Due to the convolution operation, the output spectrum \( \tilde{R}(f) \) will be wider than the input spectrum \( \tilde{S}(f) \). This broadening of the transmitted signal spectrum is caused by the channel time variations and is called frequency spreading or Doppler spreading.
  
  - If the maximum Doppler frequency \( f_m \) is much less than the signal bandwidth \( W_c \), then the Doppler spreading will not distort \( \tilde{S}(f) \).
  
  - Fortunately, this is often the case.
- The **time-variant channel transfer function** is obtained by taking the Fourier transform of the time-variant channel impulse response \( g(t, \tau) \) with respect to the delay variable \( \tau \), i.e.,

\[
T(t, f) = g(t)e^{-j2\pi f \hat{\tau}}.
\]

- Since the magnitude response is \(|T(t, f)| = |g(t)|\), all frequency components in the received signal are subject to the same time-variant amplitude gain \(|g(t)|\) and phase response \( \angle g(t) = -2\pi f \hat{\tau} \).

- The received signal is said to exhibit **“flat fading,”** because the magnitude of the time-variant channel transfer function \(|T(t, f)|\) is constant (or flat) with respect to frequency variable \( f \).

- The phase response \( \angle g(t) = -2\pi f \hat{\tau} \) is linear in \( f \) meaning that the channel simply delays the input signal and attenuates it.
Invoking the Central Limit Theorem

• Consider the transmission of an unmodulated carrier, $\tilde{s}(t) = 1$.

• For flat fading channels, the received band-pass signal has the quadrature representation

$$r(t) = g_I(t) \cos 2\pi f_c t - g_Q(t) \sin 2\pi f_c t$$

where

$$g_I(t) = \sum_{n=1}^{N} C_n \cos \phi_n(t)$$
$$g_Q(t) = \sum_{n=1}^{N} C_n \sin \phi_n(t)$$

and where $\phi_n(t) = \phi_n - 2\pi c \tau_n / \lambda_c + 2\pi f_{D,n} t$.

• The phases $\phi_n$ are independent and uniform on the interval $[-\pi, \pi)$, and the path delays $\tau_n$ are all independent with $f_c \tau_n \gg 1$. Therefore, the phases $\phi_n(t)$ at any time $t$ can be treated as being independent and uniformly distributed on the interval $[-\pi, \pi)$.

• In the limit $N \to \infty$, the central limit theorem can be invoked and $g_I(t)$ and $g_Q(t)$ can be treated as “Gaussian random processes,” i.e., at any time $t$, $g_I(t)$ and $g_Q(t)$ are Gaussian random variables.

• The “complex faded envelope” is

$$g(t) = g_I(t) + jg_Q(t)$$
Rayleigh Fading

For some types of scattering environments, \( g_I(t) \) and \( g_Q(t) \) at any time \( t_1 \) are independent identically distributed Gaussian random variables with zero mean and identical variance \( b_0 = E[g_I^2(t_1)] = E[g_Q^2(t_1)] \). This typically occurs in a rich scattering environment where there is no line-of-sight or strong specular component in the received signal (i.e., there is no dominant \( C_n \)) and isotropic antennas are used. Under such conditions, the channel exhibits Rayleigh fading.

The probability density function the envelope \( \alpha = |g(t_1)| = \sqrt{g_I^2(t_1) + g_Q^2(t_1)} \) can be obtained by using a bi-variate transformation of random variables (see Appendix in textbook).

The envelope \( \alpha = |g(t_1)| = \sqrt{g_I^2(t_1) + g_Q^2(t_1)} \) is Rayleigh distributed at any time \( t_1 \), i.e.,

\[
p_\alpha(x) = \frac{x}{b_0} \exp \left\{ -\frac{x^2}{2b_0} \right\} = \frac{2x}{\Omega_p} \exp \left\{ -\frac{x^2}{\Omega_p} \right\}, \quad x \geq 0,
\]

where \( \Omega_p = E[\alpha^2] = E[g_I^2(t_1)] + E[g_Q^2(t_1)] = 2b_0 \) is the average envelope power.

The squared-envelope \( \alpha^2 \) at any time \( t_1 \) has the exponential distribution

\[
p_{\alpha^2}(x) = \frac{1}{\Omega_p} \exp \left\{ -\frac{x}{\Omega_p} \right\}, \quad x \geq 0.
\]
A line-of-sight (LoS) or specular (strong reflected) component arrives at angle $\theta_0$. 
• For scattering environments that have a specular or LoS component, \( g_I(t) \) and \( g_Q(t) \) are Gaussian random processes with non-zero means \( m_I(t) \) and \( m_Q(t) \), respectively.

• If we again assume that \( g_I(t_1) \) and \( g_Q(t_1) \) at any time \( t_1 \) are independent random variables with variance \( b_0 = \text{E}[(g_I(t_1) - m_I(t_1))^2] = \text{E}[(g_Q(t_1) - m_Q(t_1))^2] \), then the magnitude of the envelope \( \alpha = |g(t_1)| \) at any time \( t_1 \) has a Rice distribution.

• With Aulin’s Ricean fading model

\[
\begin{align*}
m_I(t) &= \text{E}[g_I(t)] = s \cdot \cos(2\pi f_m \cos(\theta_0) t + \phi_0) \\
m_Q(t) &= \text{E}[g_Q(t)] = s \cdot \sin(2\pi f_m \cos(\theta_0) t + \phi_0)
\end{align*}
\]

where \( f_m \cos(\theta_0) \) and \( \phi_0 \) are the Doppler shift and random phase offset associated with the LoS or specular component, respectively.

• The envelope \( \alpha(t) = |g(t)| = \sqrt{g_I^2(t) + g_Q^2(t)} \) has the Rice distribution

\[
p_\alpha(x) = \frac{x}{b_0} \exp \left\{ -\frac{x^2 + s^2}{2b_0} \right\} I_o \left( \frac{xs}{b_0} \right), \quad x \geq 0
\]

- \( s^2 = m_I(t)^2 + m_Q(t)^2 \) is the specular power.

- \( 2b_0 \) is the scatter power.

- The Rice factor, \( K = s^2/2b_0 \), is the ratio of the power in the specular and scatter components.
• The average envelope power is $E[\alpha^2] = \Omega_p = s^2 + 2b_0$ and

$$s^2 = \frac{K\Omega_p}{K + 1}, \quad 2b_0 = \frac{\Omega_p}{K + 1}$$

Hence,

$$p_\alpha(x) = \frac{2x(K + 1)}{\Omega_p} \exp \left\{ -K - \frac{(K + 1)x^2}{\Omega_p} \right\} I_o \left( 2x \sqrt{\frac{K(K + 1)}{\Omega_p}} \right), \quad x \geq 0$$

• The squared-envelope $\alpha^2(t)$ has **non-central chi-square distribution** with two degrees of freedom

$$p_{\alpha^2}(x) = \frac{(K + 1)}{\Omega_p} \exp \left\{ -K - \frac{(K + 1)x}{\Omega_p} \right\} I_o \left( 2 \sqrt{\frac{K(K + 1)x}{\Omega_p}} \right), \quad x \geq 0$$

• The squared-envelope is important for the performance analysis of digital communication systems because it is proportional to the received signal power and, hence, the received signal-to-noise ratio.
The Rice distribution for several values of $K$ with $\Omega_p = 1$. 
Nakagami Fading

- Nakagami fading describes the magnitude of the received complex envelope by the distribution

\[ p_\alpha(x) = \frac{2m^m x^{2m-1}}{\Gamma(m) \Omega_p^m} \exp \left\{ -\frac{m x^2}{\Omega_p} \right\} \quad m \geq \frac{1}{2} \]

- When \( m = 1 \), the Nakagami distribution becomes the Rayleigh distribution, when \( m = 1/2 \) it becomes a one-sided Gaussian distribution, and when \( m \to \infty \) the distribution approaches an impulse (no fading).

- The Rice distribution can be closely approximated with a Nakagami distribution by using the following relation between the Rice factor \( K \) and the Nakagami shape factor \( m \)

\[ K \approx \sqrt{m^2 - m + m - 1} \]
\[ m \approx \frac{(K + 1)^2}{(2K + 1)} \]

- The squared-envelope has the Gamma distribution

\[ p_{\alpha^2}(x) = \left( \frac{m}{2\Omega_p} \right)^m \frac{x^{m-1}}{\Gamma(m)} \exp \left\{ -\frac{m x}{2\Omega_p} \right\} \]
The Nakagami pdf for several values of $m$ with $\Omega_p = 1$. 
Comparison of the cdf of the squared-envelope with Ricean and Nakagami fading.