EE6604
Personal & Mobile Communications

Week 5

Level Crossing Rate and Average Fade Duration

Statistical Channel Modeling

Fading Simulators
Level Crossing Rate and Average Fade Duration

• The **level crossing rate** (LCR) at a specified envelope level $R, L_R$, is defined as the rate (in crossings per second) at which the envelope $\alpha$ crosses the level $R$ in the positive going direction.
  
  – The LCR can be used to estimate velocity, and velocity can be used for radio resource management.

• The **average fade duration** (AFD) is the average duration that the envelope remains below a specified level $R$.
  
  – An outage occurs when the envelope fades below a critical level for a long enough period such that receiver synchronization is lost. Longer fades are usually the problem.
  
  – The probability distribution of fade durations, if it exists, would allow us to calculate probability of outage.

• Both the LCR and AFD are second-order statistics that depend on the mobile station velocity, as well as the scattering environment.

• The LCR and AFD have been derived by Rice (1948) in the context of a sinusoid in narrow-band Gaussian noise.
Rayleigh faded envelope with 2-D isotropic scattering.
Level Crossing Rate

- Obtaining the level crossing rate requires the joint pdf, \( p(\alpha, \dot{\alpha}) \), of the envelope level \( \alpha = |g(t_1)| \) and the envelope slope \( \dot{\alpha} = \frac{d|g(t_1)|}{dt} \) at any time instant \( t_1 \). Note we drop the time index \( t \) for convenience.

- In terms of \( p(\alpha, \dot{\alpha}) \), the expected amount of time the envelope lies in the interval \((R, R + d\alpha)\) for a given envelope slope \( \dot{\alpha} \) and time increment \( dt \) is

\[
p(R, \dot{\alpha})d\alpha d\dot{\alpha}dt
\]

- The time required for the envelope \( \alpha \) to traverse the interval \((R, R + d\alpha)\) “once” for a given envelope slope \( \dot{\alpha} \) is

\[
d\alpha/\dot{\alpha}
\]

- The ratio of the above two quantities is the expected number of crossings of the envelope \( \alpha \) within the interval \((R, R + d\alpha)\) for a given envelope slope \( \dot{\alpha} \) and time duration \( dt \), i.e.,

\[
\dot{\alpha}p(R, \dot{\alpha})d\alpha dt
\]
• The expected number of crossings of the envelope level \( R \) for a given envelope slope \( \dot{\alpha} \) in a time interval of duration \( T \) is

\[
\int_{0}^{T} \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha} dt = \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha} T
\]

• The expected number of crossings of the envelope level \( R \) with a positive slope in the time interval \( T \) is

\[
N_{R} = T \int_{0}^{\infty} \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha}.
\]

• Finally, the expected number of crossings of the envelope level \( R \) per second, or the level crossing rate, is obtained by dividing by the length of the interval \( T \) as

\[
L_{R} = \int_{0}^{\infty} \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha}
\]

• This is a general result that applies to any random process characterized by the joint pdf \( p(\alpha, \dot{\alpha}) \).
• Rice (BSTJ, 1948) derived the joint pdf $p(\alpha, \dot{\alpha})$ for a sine wave plus Gaussian noise. A Rician fading channel can be thought of LoS or specular (sine wave) component plus a scatter (Gaussian noise) component. For the case of a Rician fading channel,

$$p(\alpha, \dot{\alpha}) = \frac{\alpha (2\pi)^{-3/2}}{\sqrt{Bb_0}} \int_{-\pi}^{\pi} d\theta \times \exp\left\{-\frac{1}{2Bb_0}\left[B \left(\alpha^2 - 2\alpha s \cos \theta + s^2\right) + (b_0 \dot{\alpha} + b_1 s \sin \theta)^2\right]\right\}$$

where $s$ is the non-centrality parameter in the Rice distribution, and $B = b_0 b_2 - b_1^2$, where $b_0$, $b_1$, and $b_2$ are constants that depend on the scattering environment.

• Suppose that the specular or LoS component of the complex envelope $g(t)$ has a Doppler frequency equal $f_q = f_m \cos \theta_0$, where $0 \leq |f_q| \leq f_m$. Then

$$b_n = (2\pi)^n \int_{-f_m}^{f_m} S_{gg}^c(f)(f - f_q)^n df = (2\pi)^n b_0 \int_0^{2\pi} \hat{p}(\theta) G(\theta) (f_m \cos \theta - f_q)^n d\theta$$

where $\hat{p}(\theta)$ is the azimuth distribution (pdf) of the scatter component, $G(\theta)$ is the antenna gain pattern, and $S_{gg}^c(f)$ is the corresponding continuous portion of the Doppler power spectrum.

- Note that the pdf $\hat{p}(\theta)$ in this case integrates to unity.
• Note that $S_{gg}^c(f)$ is given by the Fourier transform of $\phi_{gg}^c(\tau) = \phi_{g_1g_1}^c(\tau) + j\phi_{g_1g_Q}^c(\tau)$ where

$$\phi_{g_1g_1}^c(\tau) = \frac{\Omega_p}{2} \int_0^{2\pi} \cos(2\pi f_m \tau \cos \theta) \hat{p}(\theta) G(\theta) d\theta$$

$$\phi_{g_1g_Q}^c(\tau) = \frac{\Omega_p}{2} \int_0^{2\pi} \sin(2\pi f_m \tau \cos \theta) \hat{p}(\theta) G(\theta) d\theta$$

• In some special cases, the psd $S_{gg}^c(f)$ is symmetrical about the frequency $f_q = f_m \cos \theta_0$. This condition occurs, for example, when $f_q = 0$ ($\theta_0 = 90^\circ$) and $\hat{p}(\theta) = 1/(2\pi), -\pi \leq \theta \leq \pi$.

  – Specular component arrives perpendicular to direction of motion and scatter component is characterized by 2-D isotropic scattering.

  – In this case, $b_n = 0$ for all odd values of $n$ (and in particular $b_1 = 0$) so that the joint pdf $p(\alpha, \dot{\alpha})$ reduces to the convenient product form

$$p(\alpha, \dot{\alpha}) = \sqrt{\frac{1}{2\pi b_2}} \exp \left\{ -\frac{\dot{\alpha}^2}{2b_2} \right\} \cdot \frac{\alpha}{b_0} \exp \left\{ -\frac{(\alpha^2 + s^2)}{2b_0} \right\} I_0 \left( \frac{\alpha s}{b_0} \right)$$

$$= p(\dot{\alpha}) \cdot p(\alpha) .$$

  – Since $p(\alpha, \dot{\alpha}) = p(\dot{\alpha}) \cdot p(\alpha)$, it follows that $\alpha$ and $\dot{\alpha}$ are independent for this special case.
• When $f_q = 0$ and $\hat{p}(\theta) = 1/(2\pi)$, a closed form expression can be obtained for the envelope level crossing rate.

• We have that
  
  $$b_n = \begin{cases} 
  b_0(2\pi f_m)^\frac{n-1}{2} \cdot \frac{3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} & \text{n even} \\
  0 & \text{n odd} 
  \end{cases}$$

• Therefore, $b_1 = 0$ and $b_2 = b_0(2\pi f_m)^2/2$, and

  $$L(R) = \sqrt{2\pi(K+1)}f_m\rho e^{-K-(K+1)\rho^2} I_0\left(2\rho\sqrt{K(K+1)}\right)$$

  where

  $$\rho = \frac{R}{\sqrt{\Omega_p}} = \frac{R}{R_{\text{rms}}}$$

  and $R_{\text{rms}} \triangleq \sqrt{\mathbb{E}[\alpha^2]}$ is the $rms$ envelope level.

• Under the further condition that $K = 0$ (Rayleigh fading)

  $$L(R) = \sqrt{2\pi}f_m\rho e^{-\rho^2}$$

• Notice that the level crossing rate is directly proportional to the maximum Doppler frequency $f_m$ and, hence, the MS speed $v = f_m \lambda_c$. 

Normalized level crossing rate for Rician fading. A specular component arrives with angle $\theta_0 = 90^\circ$ and there is 2-D isotropic scattering of the scatter component.
Average Fade Duration

• No known probability distribution exists for the duration of fades; this is a long standing open problem! Therefore, we consider the “average fade duration”.

• Consider a very long time interval of length $T$, and let $t_i$ be the duration of the $i$th fade below the level $R$.

• The probability that the received envelope $\alpha$ is less than $R$ is

$$\text{Pr}[\alpha \leq R] = \frac{1}{T} \sum_i t_i$$

• The average fade duration is equal to

$$\bar{t} = \frac{\text{total length of time in duration } T \text{ that the envelope is below level } R}{\text{average number of crossings in duration } T}$$

$$= \frac{\sum_i t_i}{TL(R)} = \frac{\text{Pr}[\alpha \leq R]}{L(R)}$$
If the envelope is Rician distributed, then

\[
Pr[\alpha \leq R] = \int_0^R p(\alpha) d\alpha = 1 - Q\left(\sqrt{2K}, \sqrt{2(K + 1)\rho^2}\right)
\]

where \(Q(a, b)\) is the Marcum Q function.

If we again assume that \(f_q = 0\) and \(\hat{p}(\theta) = 1/(2\pi)\), we have

\[
\bar{t} = \frac{1 - Q\left(\sqrt{2K}, \sqrt{2(K + 1)\rho^2}\right)}{\sqrt{2\pi(K + 1)f_m\rho e^{-K-(K+1)\rho^2} I_0\left(2\rho\sqrt{K(K + 1)}\right)}}
\]

If we further assume that \(K = 0\) (Rayleigh fading), then

\[
P[\alpha \leq R] = \int_0^R p(\alpha) d\alpha = 1 - e^{-\rho^2}
\]

and

\[
\bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}}.
\]
Normalized average fade duration with Ricean fading.
Scattering Mechanism for Wideband Channels

Concentric ellipses model for frequency-selective fading channels.

- Frequency-selective (wide-band) channels have strong scatterers that are located on several ellipses such that the corresponding differential path delays $\tau_i - \tau_j$ for some $i, j$, are significant compared to the modulated symbol period $T$. 
Transmission Functions

- Multipath fading channels are time-variant linear filters, whose inputs and outputs can be described in the time and frequency domains.

- There are four possible transmission functions
  - Time-variant channel impulse response $g(t, \tau)$
  - Output Doppler spread function $H(f, \nu)$
  - Time-variant transfer function $T(f, t)$
  - Doppler-spread function $S(\tau, \nu)$
Time-variant channel impulse response, $g(t, \tau)$

- Also known as the input delay spread function.
- The time varying complex channel impulse response relates the input and output time domain waveforms

$$\tilde{r}(t) = \int_0^t g(t, \tau) \tilde{s}(t - \tau) d\tau$$

- In physical terms, $g(t, \tau)$ can be interpreted as the channel response at time $t$ due to an impulse applied at time $t - \tau$. Since a physical channel is causal, $g(t, \tau) = 0$ for $\tau < 0$ and, therefore, the lower limit of integration in the convolution integral is zero.
- The convolution integral can be approximated in the discrete form

$$\tilde{r}(t) = \sum_{m=0}^{n} g(t, m\Delta\tau) \tilde{s}(t - m\Delta\tau) \Delta\tau$$

Discrete-time tapped delay line model for a multipath-fading channel.
Transfer Function, $T(f, t)$

- The transfer function relates the input and output frequencies:
  \[ \tilde{R}(f) = \tilde{S}(f)T(f, t) \]

- By using an inverse Fourier transform, we can also write
  \[ \tilde{r}(t) = \int_{-\infty}^{\infty} \tilde{S}(f)T(f, t)e^{j2\pi ft} df \]

- The time-varying channel impulse response and time-varying channel transfer function are related through the Fourier transform:
  \[ g(t, \tau) \leftrightarrow T(f, t) \]

  - Note: the Fourier transform pair is with respect to the time-delay variable $\tau$. The Fourier transform of $g(t, \tau)$ with respect to the time variable $t$ gives the Doppler spread function $S(\tau, \nu)$, i.e.,
  \[ g(t, \tau) \leftrightarrow S(\tau, \nu) \]
Fourier Transforms

Fourier transform relations between the system functions.
Statistical Correlation Functions

• Similar to flat fading channels, the channel impulse response \( g(t, \tau) = g_I(t, \tau) + jg_Q(t, \tau) \) of frequency-selective fading channels can be modelled as a complex Gaussian random process, where the quadrature components \( g_I(t, \tau) \) and \( g_Q(t, \tau) \) are Gaussian random processes.

• The transmission functions are all random processes. Since the underlying process is Gaussian, a complete statistical description of these transmission functions is provided by their means and autocorrelation functions.

• Four autocorrelation functions can be defined

\[
\phi_g(t, s; \tau, \eta) = E[g^*(t, \tau)g(s, \eta)] \\
\phi_T(f, m; t, s) = E[T^*(f, t)T(m, s)] \\
\phi_H(f, m; \nu, \mu) = E[H^*(f, \nu)H(m, \mu)] \\
\phi_S(\tau, \eta; \nu, \mu) = E[S^*(\tau, \nu)S(\eta, \mu)].
\]

• Related through double Fourier transform pairs

\[
\phi_S(\tau, \eta; \nu, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_g(t, s; \tau, \eta)e^{-j2\pi(\nu t - \mu s)} dt ds \\
\phi_g(t, s; \tau, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_S(\tau, \eta; \nu, \mu)e^{j2\pi(\nu t - \mu s)} d\nu d\mu
\]
Fourier Transforms and Correlation Functions

Double Fourier transform relations between the channel correlation functions.
WSSUS Channels

- Uncorrelated scattering in both the time-delay and Doppler shift domains.
- Practical land mobile radio channels are characterized by this behavior.
- Due to uncorrelated scattering in time-delay and Doppler shift, the channel correlation functions become:

\[
\begin{align*}
\phi_g(t, t + \Delta t; \tau, \eta) &= \psi_g(\Delta t; \tau)\delta(\eta - \tau) \\
\phi_T(f, f + \Delta f; t, t + \Delta t) &= \phi_T(\Delta f; \Delta t) \\
\phi_H(f, f + \Delta f; \nu, \mu) &= \psi_H(\Delta f; \nu)\delta(\nu - \mu) \\
\phi_S(\tau, \eta; \nu, \mu) &= \psi_S(\tau, \nu)\delta(\eta - \tau)\delta(\nu - \mu).
\end{align*}
\]

- Note the singularities \(\delta(\eta - \tau)\) and \(\delta(\nu - \mu)\) with respect to the time-delay and Doppler shift variables, respectively.
- Some correlation functions are more useful than others. The most useful functions:
  - \(\psi_g(\Delta t; \tau)\): channel correlation function
  - \(\phi_T(\Delta f; \Delta t)\): spaced-time spaced-frequency correlation function
  - \(\psi_S(\tau, \nu)\): scattering function
Fourier Transforms for WSSUS Channels

\[ \psi_g(\Delta t; \tau) \]
\[ \psi_S(\nu; \tau) \]
\[ \psi_H(\nu; \Delta f) \]
\[ \phi_T(\Delta t; \Delta f) \]
Power Delay Profile

- The autocorrelation function of the time varying impulse response is
  \[ \phi_g(t, t + \Delta t, \tau, \eta) = E[g^*(t, \tau)g(t + \Delta t, \eta)] = \psi_g(\Delta t; \tau)\delta(\eta - \tau) \]
  Note the WSS assumption.
- The function \( \psi_g(0; \tau) \equiv \psi_g(\tau) \) is called the multipath intensity profile or power delay profile.
- The average delay \( \mu_\tau \) is the mean value of \( \psi_g(\tau) \), i.e.,
  \[ \mu_\tau = \frac{\int_0^\infty \tau \psi_g(\tau)d\tau}{\int_0^\infty \psi_g(\tau)d\tau} \]
- The rms delay spread \( \sigma_\tau \) is defined as the variance of \( \psi_g(\tau) \), i.e.,
  \[ \sigma_\tau = \sqrt{\frac{\int_0^\infty (\tau - \mu_\tau)^2 \psi_g(\tau)d\tau}{\int_0^\infty \psi_g(\tau)d\tau}} \]
Simulation of Multipath-Fading Channels

• Computer simulation models are needed to generate the faded envelope with the statistical properties of a chosen reference model, i.e., a specified Doppler spectrum.

• Generally there are two categories of fading channel simulation models
  – Filtered-White-Noise models that pass white noise through an appropriate filter
  – Sum-of-Sinusoids models that sum together sinusoids having different amplitudes, frequencies and phases.

• Model accuracy vs. complexity is of concern
  – It is desirable to generate the faded envelope with low computational complexity while still maintaining high accuracy with respect to the chosen reference model.
Filtered White Noise

- Since the complex faded envelope can be modelled as a complex Gaussian random process, one approach for generating the complex faded envelope is to filter a white noise process with appropriately chosen low pass filters.

![Diagram](image)

- If the Gaussian noise sources are uncorrelated and have power spectral densities of $\Omega_p/2$ watts/Hz, and the low-pass filters have transfer function $H(f)$, then

$$S_{g_1g_1}(f) = S_{gQgQ}(f) = \frac{\Omega_p}{2} |H(f)|^2$$

$$S_{g_1g_Q}(f) = 0$$

- Two approaches: IIR filtering method and IFFT filtering method.
**IIR Filtering Method**

- implement the filters in the time domain as finite impulse response (FIR) or infinite impulse response (IIR) filters. There are two main challenges with this approach.
  
  - the normalized Doppler frequency, \( \hat{f}_m = f_m T_s \), where \( T_s \) is the simulation step size, is very small.
    
    * This can be overcome with an infinite impulse response (IIR) filter designed at a lower sampling frequency followed by an interpolator to increase the sampling frequency.
  
  - The second main challenge is that the square-root of the target Doppler spectrum for 2-D isotropic scattering and an isotropic antenna is irrational and, therefore, none of the straightforward filter design methods can be applied.

    * One possibility is to use the MATLAB function `iirlpnorm` to design the required filter.
IIR Filtering Method

- Here we consider an IIR filter of order $2K$ that is synthesized as the cascade of $K$ Direct-Form II second-order (two poles and two zeroes) sections (biquads) having the form

$$H(z) = A \prod_{k=1}^{K} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}}.$$

For example, for $f_m T_s = 0.4$, $K = 5$, and an ellipsoidal accuracy of 0.01, we obtain the coefficients tabulated below

<table>
<thead>
<tr>
<th>Stage</th>
<th>Filter Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_k$</td>
</tr>
<tr>
<td>1</td>
<td>1.5806655278853</td>
</tr>
<tr>
<td>2</td>
<td>0.19859624284546</td>
</tr>
<tr>
<td>3</td>
<td>-0.6038755531625</td>
</tr>
<tr>
<td>4</td>
<td>-0.56105447536557</td>
</tr>
<tr>
<td>5</td>
<td>-0.39828788982331</td>
</tr>
<tr>
<td>A</td>
<td>0.020939537466725</td>
</tr>
</tbody>
</table>

Coefficients for $K = 5$ biquad stage elliptical filter, $f_m T_s = 0.4$, $K = 5$
Magnitude response of the designed shaping filter, $f_m T_s = 0.4$, $K = 5$. 
IFFT Filtering Method

IDFT-based fading simulator.
To implement 2-D isotropic scattering, the filter $H[k]$ can be specified as follows:

$$
H[k] = \begin{cases} 
0 & , \quad k = 0 \\
\sqrt{\frac{1}{2\pi f_m \sqrt{1-(k/(N f_m))^2}}} & , \quad k = 1, 2, \ldots, k_m - 1 \\
\sqrt{k_m \left[ \frac{\pi}{2} - \arctan \left( \frac{k_m - 1}{\sqrt{2k_m - 1}} \right) \right]} & , \quad k = k_m \\
0 & , \quad k = k_m + 1, \ldots, N - k_m - 1 \\
\sqrt{k_m \left[ \frac{\pi}{2} - \arctan \left( \frac{k_m - 1}{\sqrt{2k_m - 1}} \right) \right]} & , \quad k = N - k_m \\
\sqrt{\frac{1}{2\pi f_m \sqrt{1-(N-k)/(N f_m))^2}}} & , \quad N - k_m + 1, \ldots, N - 1
\end{cases}
$$

One problem with the IFFT method is that the faded envelope is discontinuous from one block of $N$ samples to the next.
Sum of Sinusoids (SoS) Methods - Clarke’s Model

- With \( N \) equal strength \( (C_n = \sqrt{1/N}) \) arriving plane waves

\[
g(t) = g_I(t) + jg_Q(t)
= \sqrt{1/N} \sum_{n=1}^{N} \cos(2\pi f_m t \cos \theta_n + \hat{\phi}_n) + j\sqrt{1/N} \sum_{n=1}^{N} \sin(2\pi f_m t \cos \theta_n + \hat{\phi}_n) . \quad (1)
\]

- The normalization \( C_n = \sqrt{1/N} \) makes \( \Omega_p = 1 \).

- The phases \( \hat{\phi}_n \) are independent and uniform on \([−\pi, \pi)\).

- With 2-D isotropic scattering, the \( \theta_n \) are also independent and uniform on \([−\pi, \pi)\), and are independent of the \( \hat{\phi}_n \).

- Types of SoS simulators
  - deterministic - \( \{\theta_n\} \) and \( \{\hat{\phi}_n\} \) are fixed for all simulation runs.
  - statistical - either \( \{\theta_n\} \) or \( \{\hat{\phi}_n\} \), or both, are random for each simulation run.
  - ergodic statistical - either \( \{\theta_n\} \) or \( \{\hat{\phi}_n\} \), or both, are random, but only a single simulation run is required.
Clarke’s Model - Ensemble Averages

• The statistical properties of Clarke’s model in for finite $N$ are

$$
\phi_{g_Ig_I}(\tau) = \phi_{g_Qg_Q}(\tau) = \frac{1}{2}J_0(2\pi f_m \tau)
$$

$$
\phi_{g_Ig_Q}(\tau) = \phi_{g_Qg_I}(\tau) = 0
$$

$$
\phi_{gg}(\tau) = \frac{1}{2}J_0(2\pi f_m \tau)
$$

$$
\phi_{|g|^2|g|^2}(\tau) = E[|g|^2(t)|g|^2(t + \tau)]
= 1 + \frac{N - 1}{N}J_0^2(2\pi f_m \tau)
$$

• For finite $N$, the ensemble averaged auto- and cross-correlation of the quadrature components match those of the 2-D isotropic scattering reference model.

• The squared envelope autocorrelation reaches the desired form $1 + J_0^2(2\pi f_m \tau)$ asymptotically as $N \to \infty$. 
Clarke’s Model - Time Averages

- In simulations, time averaging is often used in place of ensemble averaging. The corresponding time average correlation functions $\hat{\phi}(\cdot)$ (all time averaged quantities are distinguished from the statistical averages with a ‘^\hat{~}' ) are random and depend on the specific realization of the random parameters in a given simulation trial.

- The variances of the time average correlation functions, defined as

$$\text{Var}[\hat{\phi}(\cdot)] = E\left[\left|\hat{\phi}(\cdot) - \lim_{N \to \infty} \phi(\cdot)\right|^2\right],$$

characterizes the closeness of a simulation trial with finite $N$ and the ideal case with $N \to \infty$.

- These variances can be derived as follows:

$$\text{Var}[\hat{\phi}_{gIgI}(\tau)] = \text{Var}[\hat{\phi}_{gQgQ}(\tau)] = \frac{1 + J_0(4\pi f_m \tau) - 2J_0^2(2\pi f_m \tau)}{8N}$$

$$\text{Var}[\hat{\phi}_{gIgQ}(\tau)] = \text{Var}[\hat{\phi}_{gQgI}(\tau)] = \frac{1 - J_0(4\pi f_m \tau)}{8N}$$

$$\text{Var}[\hat{\phi}_{gg}(\tau)] = \frac{1 - J_0^2(2\pi f_m \tau)}{4N}$$
Jakes’ Deterministic Method

• To approximate an isotropic scattering channel, it is assumed that the $N$ arriving plane waves uniformly distributed in angle of incidence:

$$\theta_n = 2\pi n/N \ , \ n = 1, 2, \ldots, N$$

• By choosing $N/2$ to be an odd integer, the sum in (1) can be rearranged into the form

$$g(t) = \sqrt{\frac{1}{N}} \sum_{n=1}^{N/2-1} \left[ e^{-j(2\pi f_m t \cos \theta_n + \phi_n)} + e^{j(2\pi f_m t \cos \theta_n + \hat{\phi}_n)} \right]$$

$$+ e^{-j(2\pi f_m t + \hat{\phi}_N)} + e^{j(2\pi f_m t + \hat{\phi}_N)}$$

(2)

• The Doppler shifts progress from $-2\pi f_m \cos(2\pi/N)$ to $+2\pi f_m \cos(2\pi/N)$ as $n$ progresses from 1 to $N/2-1$ in the first sum, while in the second sum they progress from $+2\pi f_m \cos(2\pi/N)$ to $-2\pi f_m \cos(2\pi/N)$.

• Jakes uses nonoverlapping frequencies to write $g(t)$ as

$$g(t) = \sqrt{2} \sqrt{\frac{1}{N}} \sum_{n=1}^{M} \left[ e^{-j(\hat{\phi}_n + 2\pi f_m t \cos \theta_n)} + e^{j(\hat{\phi}_n + 2\pi f_m t \cos \theta_n)} \right]$$

$$+ e^{-j(\hat{\phi}_N + 2\pi f_m t)} + e^{j(\hat{\phi}_N + 2\pi f_m t)}$$

(3)

where

$$M = \frac{1}{2} \left( \frac{N}{2} - 1 \right)$$

and the factor $\sqrt{2}$ is included so that the total power remains unchanged.
• Note that (2) and (3) are not equal. In (2) all phases are independent. However, (3) implies that $\hat{\phi}_n = -\hat{\phi}_{-N/2+n}$ and $\hat{\phi}_{-n} = -\hat{\phi}_{N/2-n}$ for $n = 1, \ldots, M$. This introduces correlation into the phases

• Jakes’ further imposes the constraint $\hat{\phi}_n = -\hat{\phi}_{-n}$ and $\hat{\phi}_N = -\hat{\phi}_{-N}$ (but with further correlation introduced in the phases) to give

$$g(t) = \sqrt{\frac{2}{N}} \left\{ 2 \sum_{n=1}^{M} \cos \beta_n \cos 2\pi f_n t + \sqrt{2} \cos \alpha \cos 2\pi f_m t \right\}$$

$$+ j \left\{ 2 \sum_{n=1}^{M} \sin \beta_n \cos 2\pi f_n t + \sqrt{2} \sin \alpha \cos 2\pi f_m t \right\}$$

where

$$\alpha = \hat{\phi}_N = \beta_n = \hat{\phi}_n$$
• Time averages:

\[
<g_I^2(t)> = \frac{2}{N} \left[ 2 \sum_{n=1}^{M} \cos^2 \beta_n + \cos^2 \alpha \right]
\]

\[
= \frac{2}{N} \left[ M + \cos^2 \alpha + \sum_{n=1}^{M} \cos 2\beta_n \right]
\]

\[
<g_Q^2(t)> = \frac{2}{N} \left[ 2 \sum_{n=1}^{M} \sin^2 \beta_n + \sin^2 \alpha \right]
\]

\[
= \frac{2}{N} \left[ M + \sin^2 \alpha - \sum_{n=1}^{M} \cos 2\beta_n \right]
\]

\[
<g_I(t)g_Q(t)> = \frac{2}{N} \left[ 2 \sum_{n=1}^{M} \sin \beta_n \cos \beta_n + \sin \alpha \cos \alpha \right]
\]

• Choose the \(\beta_n\) and \(\alpha\) so that \(g_I(t)\) and \(g_Q(t)\) have zero-mean, equal variance, and zero cross-correlation.

• The choices \(\alpha = 0\) and \(\beta_n = \pi n / M\) will yield \(<g_Q^2(t)> = M/(2M + 1)\), \(<g_I^2(t)> = (M + 1)/(2M + 1)\), and \(<g_I(t)g_Q(t)> = 0\).

• Note the small imbalance in the values of \(<g_Q^2(t)>\) and \(<g_I^2(t)>\).

• The envelope power is \(<g_I^2(t)> + <g_Q^2(t)> = \Omega_p = 1\). The envelope power can be changed to any other desired value by scaling \(g(t)\), i.e., \(\sqrt{\Omega_p} g(t)\) will have envelope power \(\Omega_p\).
Typical faded envelope generated with 8 oscillators and $f_m T = 0.1$, where $T$ seconds is the simulation step size.
Auto- and Cross-correlations

• The normalized autocorrelation function is

\[ \phi_{gg}^n(\tau) = \frac{E[g^*(t)g(t+\tau)]}{E[|g(t)|^2]} \]

• With 2-D isotropic scattering

\[ \phi_{gIgI}(\tau) = \phi_{gQgQ}(\tau) = \frac{\Omega_p}{2} J_0 (2\pi f_m \tau) \]

\[ \phi_{gIgQ}(\tau) = \phi_{gQgI}(\tau) = 0 \]

• Therefore,

\[ \phi_{gg}^n(\tau) = \frac{E[g^*(t)g(t+\tau)]}{E[|g(t)|^2]} = J_0 (2\pi f_m \tau) \]
Auto- and Cross-correlations

- For Clarke’s model with angles $\theta_n$ that are independent and uniform on $[-\pi, \pi)$, the normalized autocorrelation function is

$$
\phi_{ngg}^n(\tau) = \frac{E[g^*(t)g(t+\tau)]}{E[|g(t)|^2]} = J_0(2\pi f_m \tau) .
$$

- Clark’s model with even $N$ and the restriction $\theta_n = \frac{2\pi n}{N}$, yields the normalized ensemble averaged autocorrelation function

$$
\phi_{gg}^n(\tau) = \frac{1}{2N} \sum_{n=1}^{N} \cos \left( 2\pi f_m \tau \cos \frac{2\pi n}{N} \right) .
$$

- Clark’s model with $\theta_n = \frac{2\pi n}{N}$ yields an autocorrelation function that deviates from the desired values at large lags.

- Finally, the normalized time averaged autocorrelation function for Jakes’ method is

$$
\phi_{gg}^n(t, t + \tau) = \frac{1}{2N} \left( \cos 2\pi f_m \tau + \cos 2\pi f_m (2t + \tau) \right) + \frac{1}{N} \sum_{n=1}^{M} \left( \cos 2\pi f_n \tau + \cos 2\pi f_n (2t + \tau) \right)
$$

- Jakes’ fading simulator is not stationary or even wide-sense stationary.
Autocorrelation of inphase and quadrature components obtained with Clarke’s method, using \( \theta_n = \frac{2\pi n}{N} \) and \( N = 8 \) oscillators.
Autocorrelation of inphase and quadrature components obtained with Clarke’s method, using
\[ \theta_n = \frac{2\pi n}{N} \] and \( N = 16 \) oscillators.