Comparative analysis of statistical models for the simulation of Rayleigh faded cellular channels

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Abstract—Several new “sum-of-sinusoids” models have recently been introduced for the simulation of Rayleigh fading channels. These models are statistical in nature implying that their simulation parameters such as the Doppler frequencies are random. They have been shown to accurately reproduce some of the desired statistical properties of the faded envelope such as the time auto-correlation and the level crossing rate. However, a comparative analysis of these models, hitherto scattered throughout the literature, is not available. This paper compares these models in terms of their complexity and performance. The performance assessment is based upon the variance of the time average statistical properties from their ideal ensemble averages.

Index Terms—Channel models, fading channel simulator, Rayleigh fading, sum-of-sinusoids.

I. INTRODUCTION

The successful design and testing of mobile communication systems requires a thorough understanding of the underlying radio propagation environment. This has motivated the development of standardized channel models as in [1], and methods for the software simulation of mobile radio channels. The primary goal of any channel simulation model is to reproduce the statistical properties of the real world channel as faithfully as possible. The main idea behind a class of sum-of-sinusoids (SoS) channel models [2], [3], [4], [5], is to simulate the channel as a stationary complex Gaussian random process, formed by the sum of multiple sinusoidal waveforms having frequencies, amplitudes and phases that are appropriately selected to accurately reproduce the desired statistical properties.

SoS models can be broadly categorized as either “deterministic” or “statistical”. Deterministic SoS models have fixed Doppler frequencies, amplitudes and phases for all simulation trials, thereby, leading to deterministic properties, such as the time auto-correlation, for all simulation trials. In contrast, the statistical SoS models have at least one of the parameters - Doppler frequencies, amplitudes, or phases - as random variables for each simulation trial. As a result, the simulated channels have properties that vary for each simulation trial, but converge statistically to the desired properties over a large number of simulation trials. It should be noted that if the simulation model is ergodic, the properties may converge to the desired ones in a single simulation trial also. Among the various deterministic SoS methods [2], [4], [6], Jakes’ method has been highly popular. Pop and Beaulieu suggested addition of random phases to the Jakes’ model and discussed the properties of this new model in [7]. Recently, there has been a renewed interest in the statistical or Monte Carlo (MC) SoS methods, leading to the development of new SoS methods chiefly by Zheng and Xiao [5], [8], [9], [10]. (Unless specified otherwise, the reference to a model as statistical or MC model implies that the model has both random phases and Doppler frequencies from here onwards.) Finally, it is noteworthy that the MC SoS model due to Hoeher [3] has also gained widespread acceptance.

This paper compares the statistical SoS methods in terms of their complexity and performance. Since these SoS methods converge statistically to the desired properties, it is important to determine the number of simulation trials needed to achieve a desired convergence level. This is directly related to the variation in the time average properties of a single simulation trial from the desired ensemble average properties. Hence, we use these variations as a performance metric to compare properties such as the auto-correlation, the cross-correlation and the level crossing rate for different methods. Further, we investigate the convergence of these methods with regard to their squared envelope correlation. Our analysis shows that the squared envelope correlations derived in [5], [8], [9], [10] are incorrect. Hence, we present the correct expressions for the squared envelope correlation and compare different methods on this basis. Moreover, one of the SoS models previously believed to produce the desired channel properties turns out to be non-stationary [10]. It exhibits problems in reproducing the Gaussian statistics in the complex envelope and is non-stationary in terms of the squared envelope correlation.

The remainder of this paper is organized as follows. Section II presents the mathematical reference model and its properties. Section III presents the various statistical SoS methods proposed by Clarke, Hoeher, Zheng and Xiao along with their statistical and time-average properties. Section IV presents simulation and analytical results comparing performance of these methods, while Section V presents some concluding remarks.

This work was prepared through collaborative participation in the Collaborative Technology Alliance for Communications & Networks sponsored by the U.S. Army Research Laboratory under Cooperative Agreement DAAD19-01-2-0011. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon.

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II. THE MATHEMATICAL REFERENCE MODEL

Under frequency flat fading and two-dimensional (2-D) isotropic scattering assumptions, the properties of the reference Rayleigh faded channel model have been summarized below [2, 11]:

For the complex envelope, \( g(t) = g_i(t) + jg_q(t) \), the in-phase (I) and quadrature (Q) components have a zero-mean unit variance Gaussian distribution with auto-correlation and cross-correlation properties

\[
R_{g_i/g_i}(\tau) = E[g_i(t + \tau)g_i(t)] = J_0(\omega_d \tau) \\
R_{g_i/g_q}(\tau) = 0 \\
R_{g_q/g_q}(\tau) = \frac{1}{2}E[g(t + \tau)g^*(t)] = J_0(\omega_d \tau) \\
R_{g_i^2/g_q^2}(\tau) = E[|g(t + \tau)|^2|g(t)|^2] = 4 + 4J_0^2(\omega_d \tau)
\]

where \( E \) is the statistical expectation operator, \( J_0(\cdot) \) is the zeroth order Bessel function of the first kind, and \( \omega_d \) is the maximum angular Doppler frequency induced by the received mobility in the channel.

The goal of any channel simulator should be to reproduce these desired properties. We begin our comparisons of various channel simulators by first describing the classical Clarke’s simulation model.

III. STATISTICAL SIMULATION MODELS

A. Clarke’s model

Clarke’s model defines the complex channel gain, under a narrow-band flat fading assumption, as [12]

\[
g(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \exp[j\omega_d t \cos(\alpha_n) + \phi_n]
\]

where \( N \) is the number of propagation paths and \( \phi_n \) is the random phase of the \( n \)th multipath component that is uniformly distributed over \([0, \pi)\).\(^1\) Under the 2-D isotropic scattering assumption, the angle of arrival \( \alpha_n \sim U[-\pi, \pi] \) and is independent of the \( \phi_n \)'s. For sufficiently large \( N \), the Central Limit Theorem [13] can be invoked to show that the real part \( g_i(t) = \Re\{g(t)\} \) and the imaginary part \( g_q(t) = \Im\{g(t)\} \) of the complex envelope are zero-mean Gaussian and independent. Therefore, the envelope \( |g(t)| \) is Rayleigh distributed. Based on these assumptions, the statistical properties of Clarke’s model for a finite \( N \) are given by (1), (2), and (3) [8], while the squared envelope correlation is given by

\[
R_{g_i^2/g_q^2}(\tau) = E[|g(t + \tau)|^2|g(t)|^2] = 4 + 4J_0^2(\omega_d \tau)
\]

Eqn. (6) can be derived in a manner similar to the expression derived in Appendix I. Note that for the Clarke’s model with finite \( N \), the auto- and cross-correlation of the quadrature components match those of the reference model while the squared envelope autocorrelation reaches the desired value \( 4 + 4J_0^2(\omega_d \tau) \) asymptotically as \( N \to \infty \). Also, (6) gives the correct squared envelope autocorrelation expression for finite \( N \). (the expression in [8], Eqn. (2f) is incorrect).

The time average correlations \( \hat{R}(\cdot) \) (all time averaged quantities are distinguished from the statistical averages with a ‘'•'’ here onwards) are random and depend on a specific realization of the random parameters in a given simulation trial [8]. The variances of these correlations, defined as \( \operatorname{Var}[\hat{R}(\cdot)] = \lim_{N \to \infty} \frac{1}{N} \operatorname{Var}[R(\cdot)] \), have been derived in [8] and are repeated here (after proper power normalization to make \( R_{gg}(0)=1 \) for convenience).

\[
\operatorname{Var}[\hat{R}_{g_i,g_i}(\tau)] = \frac{1}{N} \operatorname{Var}[R_{g_i,g_i}(\tau)] = \frac{1}{2} J_0(2\omega_d \tau) - J_0^2(2\omega_d \tau) \quad (7)
\]

\[
\operatorname{Var}[\hat{R}_{g_i,g_q}(\tau)] = \frac{1}{2} J_0(2\omega_d \tau) \quad (8)
\]

\[
\operatorname{Var}[\hat{R}_{g_q,g_q}(\tau)] = \frac{1}{N} \quad (9)
\]

Based on these variances, we will compare the performance of Clarke’s model for finite \( N \) with several other models.

B. Modified Hohe\'s model - Model I

Consider the channel model [14]

\[
g(t) = g_i(t) + jg_q(t) \quad (10)
\]

\[
g_i(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos(\omega_d t f_i,n + \phi_i,n) \quad (11)
\]

\[
g_q(t) = \sqrt{\frac{2}{N}} \sum_{m=1}^{N} \cos(\omega_d t f_q,m + \phi_q,m) \quad (12)
\]

\[
f_{i/q,n/m} = \sin\left(\frac{\pi}{2} \frac{u_{i/q}}{u_{n/m}}\right) \quad (13)
\]

where the Doppler frequencies \( f_{i/q,n/m} \) for the I and Q components are determined by \( u_{i/q,n/m} \sim U(0,1) \) and are mutually independent for all \( n \) and \( m \). The random phases \( \phi_{i/q,n/m} \) are mutually independent, for all \( n \) and \( m \), and are also independent of \( u_{i/q,n/m} \)'s. For convenience, the number of sinusoids in the quadrature components are set equal, i.e., \( N_i = N_q = N \). This model is derived based on the Hohe\'s model in [3] by considering only the positive Doppler frequencies in the simulation. Hence, we refer to it as the modified Hohe\'s model. The performance of this model was compared with several deterministic simulation methods in [15]. Here, we compare it against other statistical simulation models for finite \( N \) with the aid of several statistical and time average correlations, along with the variances of the latter.

The statistical averages given by (1), (2) and (3) are valid for this model, while the squared envelope correlation is (proof is analogous to the derivation given in Appendix I)

\[
R_{g_i^2/g_q^2}(\tau) = 4 + 4J_0^2(\omega_d \tau) + \frac{1}{N} J_0(2\omega_d \tau) \quad (14)
\]

which for finite \( N \) differs from the reference model. The time averaged correlations can be derived as follows (proofs are
omitted for brevity):

\[ \hat{R}_{g_i, g_i}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g_i(t + \tau)g_i(t)dt \]
\[ = \frac{1}{N} \sum_{n=1}^{N} \cos(\omega_d f_i, n, \tau) \]
(15)

\[ \hat{R}_{g_i g_i}(\tau) = \frac{1}{N} \sum_{n=1}^{N} \cos(\omega_d f_i, n, \tau) \]
(16)

\[ \hat{R}_{g_i g_i}(\tau) = \hat{R}_{g_i g_i}(\tau) = 0 \]
(17)

\[ \hat{R}_{g g}(\tau) = \frac{1}{2} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t + \tau)g^*(t)dt \]
\[ = \frac{1}{2N} \sum_{n=1}^{N} \cos(\omega_d f_i, n, \tau) + \cos(\omega_d f_q, n, \tau) \]
(18)

Based on these expressions, the variance of the time averaged properties can be calculated, the results of which are stated in the following theorem:

**Theorem 1:** The variances of the auto-correlation and cross-correlation of the quadrature components, and the variance of the auto-correlation of the complex envelope \( g(t) \), as defined by Model I in (10)-(13), are

\[ \text{Var}[R_{g_i g_i}(\tau)] = \text{Var}[R_{g_i g_i}(\tau)] \]
\[ = \frac{1 + J_0(2\omega_d \tau) - 2J_0^2(\omega_d \tau)}{2N} \]
(19)

\[ \text{Var}[R_{g_i g_q}(\tau)] = \text{Var}[R_{g_i g_q}(\tau)] = 0 \]
(20)

\[ \text{Var}[R_{g g}(\tau)] = \frac{1 + J_0(2\omega_d \tau) - 2J_0^2(\omega_d \tau)}{4N} \].
(21)

**Proof:** Omitted for brevity. \[ \blacksquare \]

C. Zheng and Xiao’s models

Recently, Zheng and Xiao have proposed several new statistical models to simulate Rayleigh fading channels [5], [8], [9], [10]. These models differ from one another in terms of the model parameters which lead to differing statistical properties. We provide a detailed analysis of the statistical and time average properties of these models in the present section. We distinguish the models by naming them Model II, Model III and Model IV, with Model I being the modified Hoehler’s model.

**Model II**

From [5],

\[ g(t) = g_i(t) + jg_q(t) \]
(22)

\[ g_i(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos[\omega_d t \cos(\alpha_n) + \phi_{i,n}] \]
(23)

\[ g_q(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos[\omega_d t \sin(\alpha_n) + \phi_{q,n}] \]
(24)

\[ \alpha_n = \frac{2\pi n - \pi + \theta}{4N}, \quad n = 1, 2, \ldots, N \]  
(25)

where \( \theta \sim U[-\pi, \pi), \phi_{i,n} \sim U[-\pi, \pi) \) and \( \phi_{q,n} \sim U[-\pi, \pi) \) for all \( n \), and all values are mutually independent.

**Model III**

From [8], [9],

\[ g_i(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos[\omega_d t \cos(\alpha_n) + \phi_n] \]
(26)

\[ g_q(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \sin[\omega_d t \cos(\alpha_n) + \phi_n] \]
(27)

\[ \alpha_n = \frac{2\pi n + \theta + \phi_n}{N}, \quad n = 1, 2, \ldots, N \]

where \( \theta_n \sim U[-\pi, \pi) \) and \( \phi_n \sim U[-\pi, \pi) \) for all \( n \), and all values are mutually independent. For Model III, we also consider a slight modification\(^2\) to Zheng and Xiao’s model by using

\[ \alpha_n = \frac{2\pi n - \pi + \theta_n}{2N}, \quad n = 1, 2, \ldots, N \]
(29)

**Model IV**

From [10],

\[ g_i(t) = \frac{2}{\sqrt{N}} \sum_{n=1}^{N} \cos(\xi_n) \cos[\omega_d t \cos(\alpha_n) + \phi] \]
(30)

\[ g_q(t) = \frac{2}{\sqrt{N}} \sum_{n=1}^{N} \sin(\xi_n) \cos[\omega_d t \cos(\alpha_n) + \phi] \]
(31)

\[ \alpha_n = \frac{2\pi n - \pi + \theta}{4N}, \quad n = 1, 2, \ldots, N \]
(32)

where \( \xi_n \sim U[-\pi, \pi], \theta \sim U[-\pi, \pi], \) and \( \phi \sim U[-\pi, \pi] \) with all variables being mutually independent. For a sufficiently large number of sinusoids, \( N \) all of the above models produce Gaussian quadrature components and, hence, a Rayleigh faded envelope [5], [8], [10]. However, a closer inspection of Model IV reveals that the probability density function (pdf) of the quadrature components is non-stationary as stated below:

**Theorem 2:** The pdfs of the quadrature components of the complex envelope \( g(t) \) produced by Model IV are non-stationary, i.e., they are a function of time \( t \) and tend to the desired Gaussian distribution only for sufficiently large time \( t \), even in the asymptotic case when \( N \to \infty \).

**Proof:** Consider the in-phase component \( g_i(t) \) of the complex envelope given in (30). Assume that \( \theta \) and \( \phi \) are known. Then, for \( N \) sufficiently large, the application of the Central Limit Theorem [13] to \( x=g_i(t) \) (assume time \( t \) is fixed)

\(^2\)It has recently come to our notice that this model has been proposed earlier in a different form (even before Zheng and Xiao’s independent and original models) in [16]. Hence, we refer the interested reader to [16] for further details on the model and its performance.
shows that the distribution of $x$ conditioned on $\phi$ and $\theta$ is zero-mean Gaussian with variance $\sigma^2$, i.e.,

$$f_X(x|\theta, \phi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$ \hspace{1cm} (33)

$$\sigma^2 = \frac{2}{N} \sum_{n=1}^{N} \cos^2[\omega_d t \cos(\alpha_n) + \phi]$$ \hspace{1cm} (34)

$$\sigma^2 \xrightarrow{N \to \infty} 1 + J_0(2\omega_d t) \cos(2\phi)$$ \hspace{1cm} (35)

where we use [17, pp. 414 (16), pp. 415 (19)] to derive (35). Here, $H_0(\cdot)$ is the zeroth order Struve function. Now, since $\sigma^2$ is a function of $\phi$ and/or $\theta$, it is a random variable. Moreover, it depends on the time variable $t$ which suggests that the pdf $f_X(x)$ is non-Gaussian and time varying. Although a closed form expression for $f_X(x)$ cannot be obtained, it can be evaluated numerically by averaging the conditional pdf $f_X(x|\theta, \phi)$ over $\theta$ and $\phi$, i.e.,

$$f_X(x) = \int_{\theta} \int_{\phi} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)f_{\theta,\phi}(\theta, \phi) d\theta d\phi$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} \frac{1}{\sqrt{\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) d\theta d\phi$$ \hspace{1cm} (36)

where we used the fact that $\theta$ and $\phi$ are mutually independent and uniformly distributed over $[-\pi, \pi)$. Using this equation with $N=50$, we evaluate the pdf of $x$ at different times $t$ and plot the results in Fig. 1. To facilitate comparison, the pdf of a standard Gaussian random variable $N(0, 1)$ is also plotted in Fig. 1. The pdf is clearly time varying and approaches the desired pdf only for sufficiently large $t$. Simulation results, though not presented in Fig. 1 to retain clarity, also confirm this fact. This proves the non-stationary nature of the pdf of the in-phase component (and analogously of the quadrature component) of the complex envelope $g(t)$.  

Owing to the non-stationary, non-Gaussian, nature exhibited by Model IV, it is not a good candidate for channel simulation purposes. Model IV is also non-stationary with respect to the properties of the squared envelope as discussed below. The statistical correlation functions of the quadrature components for Models II, III, and IV have been computed by Zheng and Xiao and match the desired functions in (1), (2), and (3). The time average correlations are easy to derive and, hence, are not provided here. Besides, unlike the earlier methods, closed form expressions for the variances of the time average correlations do not exist for Zheng and Xiao’s models. The variances will be computed via simulations, as described in section IV. Here, we only present the statistical squared envelope correlation expressions since they represent the corrected versions of expressions given in [5] (eqn. 5f), [8] (eqn. 8f) and [10] (eqn. 16f) for Models II, III, and IV, respectively.

**Theorem 3:** The auto-correlation functions for the squared envelope obtained with Models II, III and IV are given by (37), (38), and (39) given at the bottom of the page, respectively.

**Proof:** Detailed derivations of (37), (38) and (39) are omitted for brevity. A brief outline of the proof for (38) and

\[ R_{|g|^2|g|^2}(\tau) = 4 + \frac{J_0(2\omega_d \tau)}{N} + \frac{4}{N^2} \sum_{n=1}^{N} \sum_{m(\neq n) = 1}^{N} E\{ \cos[\omega_d \tau \cos(\alpha_n)] \cos[\omega_d \tau \cos(\alpha_m)] \} \] \hspace{1cm} (37)

\[ R_{|g|^2|g|^2}(\tau) = 4 + 4J_0^2(\omega_d \tau) - \frac{4}{N^2} \sum_{n=1}^{N} \left[ \left( \left[ E\{ \cos[\omega_d \tau \cos(\alpha_n)] \} \right]^2 + \left[ E\{ \sin[\omega_d \tau \cos(\alpha_n)] \} \right]^2 \right) \right] \] \hspace{1cm} (38)

\[ R_{|g|^2|g|^2}(\tau) = 4 + \frac{2J_0(2\omega_d \tau)}{N} + \frac{4}{N^2} \sum_{n=1}^{N} \sum_{m(\neq n) = 1}^{N} E\{ \cos[\omega_d \tau \cos(\alpha_n)] \cos[\omega_d \tau \cos(\alpha_m)] \} \]

\[ + \frac{2}{N^2} \sum_{n=1}^{N} \sum_{m(\neq n) = 1}^{N} E\{ \cos[2\omega_d (t + \tau) \cos(\alpha_n) - 2\omega_d t \cos(\alpha_m)] \} \]

\[ + \frac{2}{N^2} \sum_{n=1}^{N} \sum_{m(\neq n) = 1}^{N} E\{ \cos[\omega_d (2t + \tau) \cos(\alpha_n) - \omega_d (2t + \tau) \cos(\alpha_m)] \} \] \hspace{1cm} (39)
Fig. 2. Auto-correlation of the squared envelope generated by Model IV as a function of time.

### TABLE I

<table>
<thead>
<tr>
<th>Model</th>
<th>Computations needed to generate 1 sample of $g(t)$</th>
<th>Relative simulation time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clarke</td>
<td>$2N$ cosine, $N$ additions</td>
<td>$T_s$</td>
</tr>
<tr>
<td>Model I</td>
<td>$2N$ cosine, $2N$ additions</td>
<td>$1.20T_s$</td>
</tr>
<tr>
<td>Model II</td>
<td>$2N$ cosine, $2N$ additions</td>
<td>$1.20T_s$</td>
</tr>
<tr>
<td>Model III</td>
<td>$2N$ cosine, $N$ additions</td>
<td>$T_s$</td>
</tr>
</tbody>
</table>

The number of random variables required by all models is almost the same ($2N$), except Model I which requires double the number of random variables ($4N$).

Note that there are no closed form expressions for the squared envelope correlation due to the presence of the last terms in (37), (38) and (39); they must be evaluated numerically. It is important to note that for Model IV, the squared envelope correlation is a function of both the time variable $t$ and the delay variable $\tau$, exposing the non-stationary nature of the squared envelope. This is further verified in Fig. 2, where the squared envelope correlation is plotted for different time values ($t$'s) by numerically evaluating (39), using $N = 11$. It is clear from Fig. 2 that the squared envelope correlation varies with time $t$. Owing to the non-stationarity problems exhibited by Model IV, we exclude it from further analysis.

IV. Analysis and Simulation Results

This section combines the analysis of earlier sections to compare the various simulation models. We begin by comparing their relative complexity in terms of the number of computations required and the simulation time.

A. Complexity Analysis

Table I summarizes the number of operations needed to generate one sample of the complex envelope $g(t)$ for different models. Here, we count only the frequently executed operations, neglecting operations that are performed only once during a simulation trial such as the generation of random variables. The relative simulation times needed to generate $10^5$ samples of $g(t)$ in Matlab on a Pentium III laptop are also tabulated in Table I. From the Table, it is evident that Clarke's model and Model III require the least simulation time owing to their low complexity.

B. Auto-correlation/cross-correlation analysis

We now investigate the performance of the different models in terms of their auto- and cross-correlation functions. All the results presented here are obtained using $N=11$ and a normalized sampling period $f_dT_s$ of 0.001 ($f_d$ is the maximum Doppler frequency and $T_s$ is the sampling period). The variances are computed by averaging over $10^4$ simulation trials for each value of time delay $\tau$. Fig. 3 presents the variance in

Fig. 3. Variance of the auto-correlation of the in-phase component.

Fig. 4. Variance of the cross-correlation of the quadrature components.
the auto-correlation of the in-phase component of the complex envelope $g(t)$. The quadrature component has similar variance. Fig. 4 compares the variance in the cross-correlation between the quadrature components while Fig. 5 depicts the variance in the auto-correlation of the complex envelope $g(t)$.

From Fig. 3 and Fig. 5, it is interesting to note that the variances for Model II increase for longer time delays $|\tau| \geq \frac{N(f_d)}{2}$. However, for shorter time delays (and even longer time delays, see Fig. 5), which are of more interest for most communication systems [16], the variance of correlation functions of Model II are lower than the variances for all (or most) other models. Hence, Model II is the best among all models for a finite $N$.

Moreover, Model II always produces uncorrelated in-phase and quadrature components, a property which is necessary to yield a Rayleigh distributed envelope. Interestingly, Model II performs better than Model I in spite of their same relative operational complexity. This can be attributed to the proper selection of the simulation parameters in Model II. We can also see that the modified Model III proposed here performs better than the original Model III validating our selection of the new model parameters, $\alpha_n$. Further, the statistical squared envelope correlations exhibited by Zheng and Xiao’s models are compared in Fig. 6 based on the analytical expressions derived earlier. Simulation curves are obtained by averaging the time-averaged squared envelope correlation over 50 simulation trials. Here, we do not consider their variances since neither of them gives the exact desired correlation. Instead, we look for a model which approximates the desired correlation as closely as possible in a statistical sense. From Fig. 6, we find that all the models give a fairly good approximation of the desired correlation with Model I being the most accurate.

C. Level crossing rate analysis

Here we evaluate the various models in terms of their higher order statistics. Specifically, we focus on the level crossing rate. Zheng and Xiao used simulations to evaluate the level crossing rate, while in our work we use an analytical approach. An analytical approach enables us to compare the different models easily and precisely. Under the assumption that the Gaussian approximation of the channel holds even with a finite number of sinusoids, we determine the LCR by adopting the analysis in [14]. Since the LCR for each simulation trial varies randomly, we quantify the variance in the error between the LCR produced by a simulation trial and the desired LCR. Before proceeding further, a note on the notation used. We need to consider two cases separately as described below:

Case I: The quadrature components of $g(t)$ are uncorrelated, i.e., the time average correlation $\hat{R}_{gg}(\tau) = 0$ for all $\tau$, and $\beta_i \neq \beta_q$, where

$$\hat{\beta}_i = - \left[ \frac{d^2 \hat{R}_{gg_i}(\tau)}{d\tau^2} \right]_{\tau=0}, \quad \hat{\beta}_q = - \left[ \frac{d^2 \hat{R}_{gg_q}(\tau)}{d\tau^2} \right]_{\tau=0} \quad (40)$$

Then, the LCR at signal envelope level $R$ is [14]

$$\hat{L}_R = \sqrt{\frac{\hat{\beta}_i}{2\pi} E(\hat{k})} P_R(r) \quad (41)$$

$$\hat{k} = \sqrt{\frac{\hat{\beta}_i - \beta_q}{\beta_i}} \hat{\beta}_i \geq \hat{\beta}_q \quad (42)$$

where $E$ is the complete elliptic integral of the second kind and $P_R(r)$ represents the cumulative distribution function of the envelope $|g(t)| = R$. The desired LCR value at level $R$ is obtained by substituting the statistical averaged quantities in the above equation by using $\hat{\beta}_i = \hat{\beta}_q = \beta$. Then, the error in the simulated LCR is

$$e_R = \frac{L_R - \hat{L}_R}{L_R} = 1 - \sqrt{\frac{\hat{\beta}_i}{\beta}} \left( \frac{2}{\pi} \right) E(\hat{k}) \quad (43)$$

This analysis is valid for Models I and II since they satisfy the assumptions made earlier.

Case II: The quadrature components of $g(t)$ are correlated,

\footnote{This can be easily verified by computing time average auto-correlations for different models.}
i.e., the time average correlation $\hat{R}_{q_k q_k}(\tau) \neq 0$ for all $\tau$, and $\hat{\beta}_i = \hat{\beta}_q = \hat{\beta}$. Then,

$$\hat{L}_R = \sqrt{b_0 b_2 - b_1^2} \overline{P_R(R)}$$

$$e_R = 1 - \sqrt{b_0 b_2 - b_1^2}$$

where $b_0$, $b_1$, and $b_2$ can be shown to be

$$b_n = (j)^n \left[ \frac{d^n \hat{R}_{q q}(\tau)}{d\tau^n} \right]_{\tau=0},$$

by using the properties of Fourier Transforms and the definitions provided in [11]. Case II holds for the Clarke’s model and Model III since they generate correlated quadrature components.

Using this analysis, we compute the errors in the LCRs for different models via simulations assuming $N = 11$ and a maximum Doppler frequency of 100 Hz and find the variance of the error for different models (averaged over 50000 simulations), which are tabulated in the Table II. We can conclude from this analysis that Model II is the best even in terms of LCR, followed by Model III.

**D. Comparison with other models**

Finally, it is worth comparing the MC models discussed here with other statistical models such as the Method of Exact Doppler Spread (MEDS) [14] which have only random phases and not random frequencies. MEDS method has found widespread acceptance since it reproduces desired autocorrelation with good accuracy. The MEDS model is ergodic and, therefore, a single simulation trial is representative of the MEDS model properties. In contrast, the MC models discussed here are complex since they require several simulation trials for convergence. However, to partly overcome the complexity problem, “multiple parameter set Monte Carlo” (MPS-MC) simulation method has been introduced in [16]. The MPS-MC method divides a simulation trial into several frames and generates random Doppler frequencies and phases for each frame. With this method, the performance of MC models is considerably improved and found to be even better than the MEDS method. Due to space constraints, we do not provide the exact details of the MEDS or the MPS-MC methods here, but refer the reader to [14] and [16], respectively. However, for completeness, we provide a brief comparison of the performance between the MEDS method and the MPS-MC method applied to Model II. Note, that MPS-MC is not applied to MEDS since its properties do not vary over simulation trials.

**TABLE II**

<table>
<thead>
<tr>
<th>Model</th>
<th>Error mean</th>
<th>Error variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clarke</td>
<td>5.4e-2</td>
<td>1.56e-2</td>
</tr>
<tr>
<td>Model I</td>
<td>4.2e-3</td>
<td>5.8e-3</td>
</tr>
<tr>
<td>Model II</td>
<td>2e-4</td>
<td>1e-8</td>
</tr>
<tr>
<td>Model III</td>
<td>2e-3</td>
<td>1.2e-3</td>
</tr>
</tbody>
</table>

For MPS-MC 10 sinusoids were used for simulation of both the I and Q-phase components. $10^6$ samples were generated by dividing them into $10^2$ frames of length $10^4$ samples each to get time average auto-correlation results. For the MEDS, the Doppler frequencies were chosen as specified in [14], the phases were chosen $U[-\pi, \pi]$, and 10 and 11 sinusoids were used for the I and Q-phase components, respectively.

Fig. 7 shows the auto-correlation of the complex envelope obtained with a single simulation trial using the two methods for $f_d T_s = 0.05$. Clearly, MPS-MC Model II provides a better fit over a wide range of time delays than the MEDS method. Results over numerous simulation trials showed the same trend. It should also be noted here that the auto-correlation with the MPS-MC model is zero if the time delay exceeds the frame length. Hence, the frame length should be sufficiently long to cover the time delays of interest to get meaningful results.
Finally, in Fig. 8, we compare the BER performance of non-coherent differential BPSK (D-BPSK) modulation. Non-coherent D-BPSK was chosen since its performance depends on the channel auto-correlation [18] and, hence, provides a better understanding as to which model is better. BER results were obtained by averaging over $10^7$ bits (samples) consisting of $10^4$ frames for the MPS-MC model for two different values of the normalized maximum Doppler frequency $f_dT_s$. Fig. 8 shows that the BER obtained using both the MPS-MC Model II and the MEDS model matches the theoretical BER [18]. Further, the undesirable variation of the BER over several simulation trials for the MC method, as proven in [19], is eliminated by the MPS-MC method as seen from the results of two simulation trials. Though 20 simulation trials were conducted to confirm this, results of all 20 trials are not shown to preserve clarity. Without further comparisons due to space constraints, we simply state that the MC models discussed in earlier sections can provide better performance compared to MEDS at the expense of somewhat increased complexity.

A detailed comparison between the MPS-MC method and the MEDS method is out of the scope of this paper and might be a topic of further study. Issues such as the effect of simulating different frames independently on channel estimation, synchronization, and Viterbi decoding should be addressed in such a study.

V. CONCLUSION

The paper has presented a rigorous analysis comparing various SoS based statistical methods available in the literature for simulating flat-faded Rayleigh mobile channels. We addressed performance as well as complexity issues to identify an appropriate model for simulation purposes. From our analysis, we can conclude that Model II, though slightly more complex than some of the other models, is the best for the purpose of simulation of Rayleigh faded mobile channels. Also, based on the allowable complexity and the desired performance, we can select other models by using the analysis presented here. In particular, the modified Model III can provide a good trade-off between complexity and performance.

VI. ACKNOWLEDGEMENT

The authors are sincerely grateful to the anonymous reviewers whose feedback helped improve the quality of the paper.

Disclaimer: The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U. S. Government

APPENDIX I
DERIVATION OF (38)

Here we derive the squared envelope correlation for Model III. Following this proof, the squared envelope correlation for other models may be derived. We follow a procedure similar to the one outlined in [10]. We have

$$R_{[g|z]|g_1^2}(\tau) = \mathbb{E}[|g(t + \tau)|^2|g(t)|^2]$$

$$= \mathbb{E}[g(t + \tau)g_1^2(t)] + \mathbb{E}[g_1^2(t + \tau)g(t)] + \mathbb{E}[g(t + \tau)g_1^2(t)] + \mathbb{E}[g_1^2(t + \tau)g(t)]$$

We compute $\mathbb{E}[g(t + \tau)g_1^2(t)]$ in this equation as follows, with the remainder of the proof following a similar pattern:

$$\mathbb{E}[|g_i(t + \tau)|^2|g_i(t)|^2]$$

$$= \frac{4}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \mathbb{E}\{ \cos[\omega_d(t + \tau)\cos(\alpha_m) + \phi_m] \times \cos[\omega_d(t + \tau)\cos(\alpha_k) + \phi_k] \times \cos[\omega_d\tau\cos(\alpha_l) + \phi_l] \}.$$ (48)

The mutual independence of the $\phi$’s ensures that all terms in the above equation are zero, except the four terms with: 1) $n = m = k = l$; 2) $n = m$, $k = l$, $n \neq k$; 3) $n = k$, $m = l$, $n \neq m$; 4) $n = l$, $m = k$, $n \neq m$. We compute each of these terms individually to derive the overall expression.\(^5\)

Term 1: $n = m = k = l$

$$= \sum_{n=1}^{N} \mathbb{E}\{ \frac{1}{2} \{ 1 + \cos[2\omega_d(t + \tau)\cos(\alpha_n) + 2\phi_n] \} \times \{ 1 + \cos[2\omega_d\tau\cos(\alpha_n)] \} \}$$

$$= \frac{1}{4} \sum_{n=1}^{N} \left( \frac{1}{2} + \mathbb{E}\{ \cos[2\omega_d\tau\cos(\alpha_n)] \} \right) = \frac{N}{4} + \frac{N}{8} N J_0(2\omega_d\tau)$$ (49)

resulting from the fact that [17, pp. 415 (18)]

$$\sum_{n=1}^{N} \mathbb{E}\{ \cos[2\omega_d\tau\cos(\alpha_n)] \} = N J_0(2\omega_d\tau).$$ (50)

Term 2: $n = m$, $k = l$, $n \neq k$

$$= \sum_{n=1}^{N} \mathbb{E}\{ \cos^2[\omega_d(t + \tau)\cos(\alpha_n) + \phi_n] \}$$

$$\times \sum_{k=1, k \neq n}^{N} \mathbb{E}\{ \cos^2[\omega_d\tau\cos(\alpha_k) + \phi_k] \}$$

$$= \sum_{n=1}^{N} \frac{1}{2} \sum_{k=1, k \neq n}^{N} \frac{1}{2} = \frac{N(N - 1)}{4}.$$ (51)

\(^5\)We apply the constant $4/N^2$ factor at the end and, hence, omit it here.
Term 3: \( n = k, m = l, n \neq m \)

\[
= \left( \sum_{n=1}^{N} E \left\{ \cos[\omega_d(t + \tau) \cos(\alpha_n) + \phi_n] \right\} \right) \\
\times \sum_{m=1, m \neq n}^{N} E \left\{ \cos[\omega_d(t + \tau) \cos(\alpha_m) + \phi_m] \right\} \\
\times \sum_{n=1}^{N} E \left\{ \cos[\omega_d(t + \tau) \cos(\alpha_n) + \phi_n] \right\} \\
\times \sum_{m=1, m \neq n}^{N} E \left\{ \cos[\omega_d(t + \tau) \cos(\alpha_m) + \phi_m] \right\}
\]

\[
= \frac{N}{4} J_0^2(\omega_d \tau) - \frac{1}{4} \sum_{n=1}^{N} E \left\{ \cos[\omega_d \cos(\alpha_n) \tau] \right\}^2 . \tag{52}
\]

Term 4 can be shown equal to Term 3. Then, adding all the four terms, we get

\[
E[g_i^2(t + \tau)g_i^2(t)] = 1 + \frac{1}{2} \sum_{n=1}^{N} J_n(2\omega_d \tau) + 2 J_0(\omega_d \tau)
\]

\[
- \frac{2}{N^2} \sum_{n=1}^{N} E \left\{ \cos[\omega_d \cos(\alpha_n) \tau] \right\}^2 . \tag{53}
\]

Similarly, we can show that

\[
E[g_i^2(t + \tau)g_i^2(t)] = E[g_i^2(t + \tau)g_i^2(t)]
\]

\[
E[g_i^2(t + \tau)g_i^2(t)] = 1 - \frac{1}{2} \sum_{n=1}^{N} J_n(2\omega_d \tau)
\]

\[
- \frac{2}{N^2} \sum_{n=1}^{N} E \left\{ \sin[\omega_d \cos(\alpha_n) \tau] \right\}^2 . \tag{55}
\]

\[
E[g_i^2(t + \tau)g_i^2(t)] = E[g_i^2(t + \tau)g_i^2(t)]. \tag{56}
\]

Adding these terms together, gives the desired expression (38), which cannot be simplified further due to the presence of the last term and has to be evaluated numerically.

**APPENDIX II**

**SQUARED ENVELOPE CORRELATION FOR MODEL IV**

Analogous to the derivation for the squared envelope correlation for Model III provided in Appendix I, we derive the squared envelope correlation for Model IV. The proof reveals the non-stationary nature of the simulation model by providing the correct correlation expression in contrast to the expression derived in [10], thereby making our analysis more concrete. Again, we consider the derivation of \( E[g_i^2(t + \tau)g_i^2(t)] \) term only. Analogous to Appendix I, Term 1 can be shown to be:

Term 1: \( n = m = l \)

\[
= \sum_{n=1}^{N} E \left\{ \cos^4(\xi_n) \cos^2[\omega_d(t + \tau) \cos(\alpha_n) + \phi] \right\} + 2 \sum_{n=1}^{N} E \left\{ \cos[\omega_d(t + \tau) \cos(\alpha_n) + \phi] \right\} \cos[\omega_d(t + \tau) \cos(\alpha_n) + \phi] \}
\]

\[
= \frac{3}{32} \left[ N + \frac{1}{2} NJ_0(2\omega_d \tau) \right] . \tag{57}
\]

Term 2: \( n = m, k = l, n \neq k \)

\[
= \sum_{n=1}^{N} \sum_{k=1, k \neq n}^{N} E \left\{ \cos^2(\xi_n) \cos^2(\xi_k) \right\} \cos[\omega_d(t + \tau) \cos(\alpha_n) + \phi] \cos[\omega_d(t + \tau) \cos(\alpha_k) + \phi]
\]

\[
= \frac{1}{16} \sum_{n=1}^{N} \sum_{k=1, k \neq n}^{N} E \left\{ 1 + \cos[\omega_d(t + \tau) \cos(\alpha_n) + 2\phi] \right\} \cos[\omega_d(t + \tau) \cos(\alpha_k) + 2\phi]
\]

\[
= \frac{N(N-1)}{16} + \frac{1}{32} \sum_{n=1}^{N} \sum_{k=1, k \neq n}^{N} E \left\{ \cos[\omega_d(t + \tau) \cos(\alpha_n) \cos(\alpha_k) + \phi] \right\} \cos[\omega_d(t + \tau) \cos(\alpha_k) + \phi] . \tag{58}
\]

Unlike the derivation of Term 2 in [10], the expectation of the product of the two cosine terms cannot be simplified into the product of the expectations of the individual cosine terms due to the presence of the random phase \( \phi \) in both these terms. This leads to the correct expression given above in contrast to the expression provided in [10]. Further, it is clear that this term depends not only on the time difference \( \tau \) but also the time variable \( t \). Hence, the squared envelope correlation is non-stationary. Following a similar methodology, we can derive other terms to obtain (39), the squared envelope correlation for Model IV. The other terms in (39) that depend on \( t \) are also non-zero, thereby, contributing to the non-stationarity of the model.

**REFERENCES**


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