ECE 6604 Assignment # 3 Solutions

1/ 2.26

a) $\psi_g(\Delta t : \tau) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \phi_T(\Delta f : \Delta t)$

Transform is w.r.t. τ or Δf

$$\frac{1}{a + j2\pi\Delta f} \longleftrightarrow e^{-a\tau} u(\tau) \quad ; \quad u(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \text{else} \end{cases}$$

Hence,

$$\psi_g(\Delta t : \tau) = e^{-b|\Delta t|} e^{-a\tau} u(\tau)$$

b) $\psi_g(\Delta t : \tau) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \psi_s(\nu ; \tau)$

Transform is w.r.t. Δt or ν

$$e^{-b|\Delta t|} \longleftrightarrow \frac{2b}{b^2 + (2\pi\nu)^2}$$

Hence,

$$\psi_s(\nu ; \tau) = \frac{2b}{b^2 + (2\pi\nu)^2} \cdot e^{-a\tau} u(\tau)$$

c) $\psi_g(\tau) = e^{-b} e^{-a\tau} u(\tau)$

$$\int_0^\infty \psi_g(\tau) d\tau = \frac{e^{-b}}{a} \qquad \int_0^\infty \tau \psi_g(\tau) d\tau = \frac{e^{-b}}{a^2}$$

$$\mu_\tau = \frac{\int_0^\infty \tau \psi_g(\tau) d\tau}{\int_0^\infty \psi_g(\tau) d\tau} = \frac{1}{a}$$

$$\sigma_\tau^2 = \frac{\int_0^\infty (\tau - \mu_\tau)^2 \psi_g(\tau) d\tau}{\int_0^\infty \psi_g(\tau) d\tau} = \frac{\int_0^\infty \tau^2 \psi_g(\tau)}{\int_0^\infty \psi_g(\tau)} - \mu_\tau^2$$

$$= \frac{2e^{-b}/a^3}{e^{-b}/a} - \frac{1}{a^2} = \frac{1}{a^2} \Rightarrow \boxed{\sigma_\tau = \frac{1}{a}}$$

2 2.32 a)

$$T(f) = \mathcal{F}_{\tau}\{g(t,\tau)\}$$

$$= \frac{1}{\sqrt{L}} \sum_{k=1}^{L} e^{-j2\pi f (k-1)\Delta_\tau}$$

$$= \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} \left(e^{-j2\pi f \Delta_\tau}\right)^{l}$$

But
$$\sum_{l=0}^{L-1} x^{l} = \frac{1-x^{L}}{1-x}$$

$$T(f) = \frac{1}{\sqrt{L}} \frac{1 - e^{-j2\pi f L \Delta_\tau}}{1 - e^{-j2\pi f \Delta_\tau}}$$

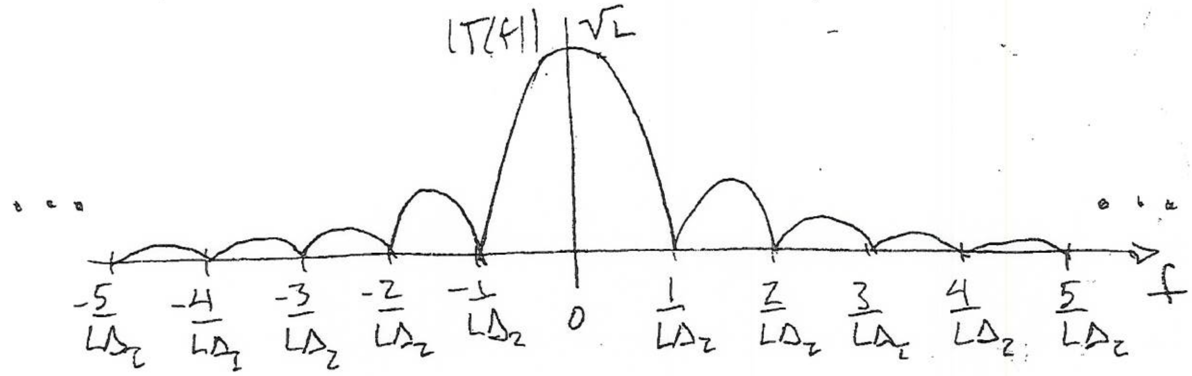
$$T(f) = \frac{1}{\sqrt{L}} \frac{e^{-j\pi f L \Delta_\tau}\left(e^{j\pi f L \Delta_\tau} - e^{-j\pi f L \Delta_\tau}\right)/j2}{e^{-j\pi f \Delta_\tau}\left(e^{j\pi f \Delta_\tau} - e^{-j\pi f \Delta_\tau}\right)/j2}$$

$$= \frac{1}{\sqrt{L}} \frac{\sin(\pi f L \Delta_\tau)}{\sin(\pi f \Delta_\tau)} e^{-j\pi f (L-1)\Delta_\tau}$$

$$= \sqrt{L}\, D_L(f)\, e^{-j\pi f (L-1)\Delta_\tau}$$

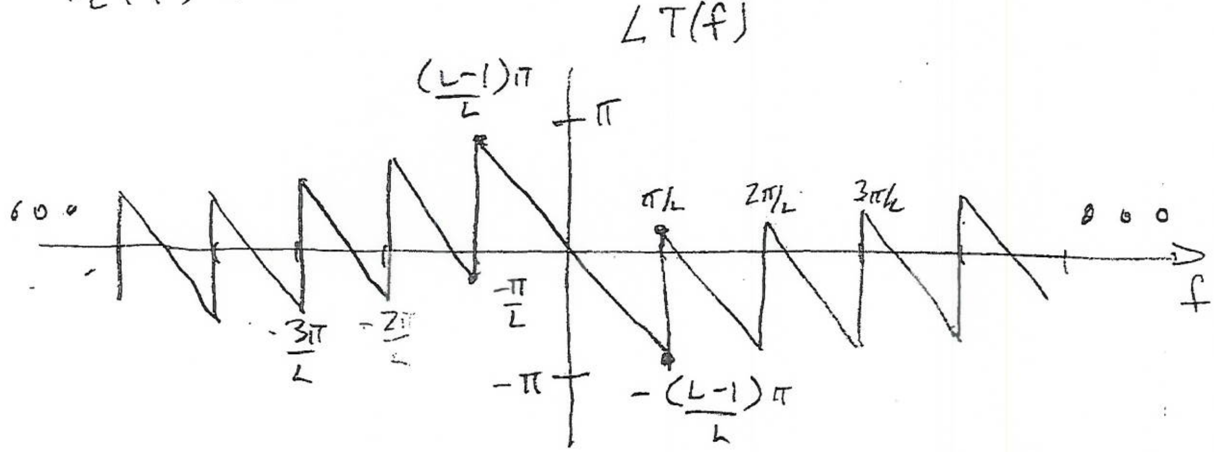
$$D_L(f) = \frac{\sin(\pi f L \Delta_\tau)}{L \sin(\pi f \Delta_\tau)} \quad \text{Dirichlet function}$$

$$|T(f)| = \sqrt{L}\, |D_L(f)|$$

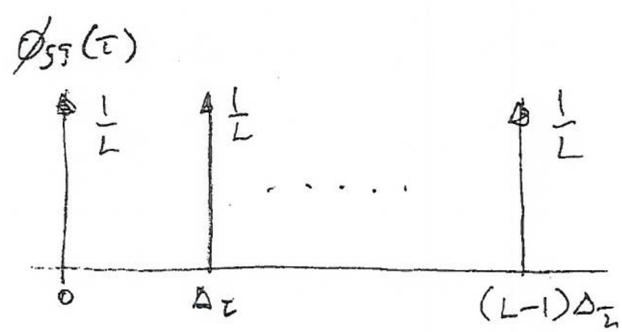


$$\angle T(f) = -\pi f (L-1)\Delta_\tau$$

However phase jumps by $\pm\pi$ when $D_L(f) < 0$.



b)



Total Power = 1

$$\mu_\tau = \frac{\sum_{i=0}^{L-1} \tau_i P_i}{\cancel{\sum_{i=0}^{L-1} P_i}} = \sum_{k=0}^{L-1} k \Delta_\tau \frac{1}{L}$$

$$= \frac{\Delta_\tau}{L} \sum_{k=1}^{L-1} k$$

$$= \frac{L(L-1)}{2} \cdot \frac{\Delta_\tau}{L}$$

$$= \frac{(L-1)\Delta_\tau}{2}$$

$\sigma_\tau^2 = E[\tau^2] - \mu_\tau^2$ since Total Power = 1

$$E[\tau^2] = \sum_{k=0}^{L-1} k^2 \Delta_\tau^2 \frac{1}{L}$$

$$= \frac{\Delta_\tau^2}{L} \sum_{k=1}^{L-1} k^2$$

$$= \frac{(L-1)\cancel{L}(2(L-1)+1)}{6} \frac{\Delta_\tau^2}{\cancel{L}}$$

$$= \frac{(L-1)(2L-1)}{6} \Delta_\tau^2$$

$$= \frac{2L^2 - 3L + 1}{6} \Delta_\tau^2$$

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$$\sigma_\tau^2 = \left(\frac{2L^2 - 3L + 1}{6} - \frac{L^2 - 2L + 1}{4} \right) \Delta_\tau^2$$

$$= \left(\frac{4L^2 - 6L + 2}{12} - \frac{3L^2 - 6L + 3}{12} \right) \Delta_\tau^2$$

$$= \left(\frac{L^2}{12} - \frac{1}{12} \right) \Delta_\tau^2 = \frac{L^2 - 1}{12} \Delta_\tau^2$$

$$\sigma_\tau = \sqrt{\frac{L^2 - 1}{12}} \Delta_\tau$$

3/ 2.36 a) Assume the strongest ray has power of 1W. This is arbitrary, and any number will work. This gives the power delay profile below

Tap	Delay (τ_k)	Power (P_k)
1	0	1.0
2	2	0.1
3	6	0.3162
4	8	0.0316

$$\mu_\tau = \sum_{k=1}^{K} \tau_k P_k \Big/ \sum_{k=1}^{4} P_k$$

$$= \frac{0 \times 1.0 + 2 \times 0.1 + 6 \times 0.3162 + 8 \times 0.0316}{1.0 + 0.1 + 0.3162 + 0.0316}$$

$$= 2.35/1.4478 = \boxed{1.623\ \mu s}$$

b)

$$\sigma_\tau = \sqrt{\frac{\sum_{k=1}^{4} (\tau_k - \mu_\tau)^2 P_k}{\sum_{k=1}^{4} P_k}}$$

$$= \sqrt{\frac{(0-\mu_\tau)^2(1.0) + (2-\mu_\tau)^2(0.1) + (6-\mu_\tau)^2(0.3162) + (8-\mu_\tau)^2 \times 0.0316}{1.0 + 0.1 + 0.3162 + 0.0316}}$$

$$= \boxed{2.627\ \mu s}$$

c) $$\psi_g(\tau) = \sum_{i=1}^{4} \Omega_i \, \delta(\tau - \tau_i)$$

$$\phi_f(\Delta_f) = \mathcal{F}\{\psi_g(\tau)\}$$

$$= \sum_{i=1}^{4} \Omega_i \, e^{-j 2\pi \Delta_f \tau_i}$$

$$= 1 + 0.1 e^{-j 4\pi \Delta_f \times 10^{-6}} + 0.3162 e^{-j 12\pi \Delta_f \times 10^{-6}} + 0.0316 e^{-j 16\pi \Delta_f \times 10^{-6}}$$

The fregueny correlation function may be noromulized so its maximum values is unity

$$\hat{\phi}_f(\Delta_f) = \frac{1}{1.4478} \left\{ 1 + 0.1 e^{-j 4\pi \Delta_f \times 10^{-6}} + 0.3162 e^{-j 12\pi \Delta_f \times 10^{-6}} + 0.0316 e^{-j 16\pi \Delta_f \times 10^{-6}} \right\}$$

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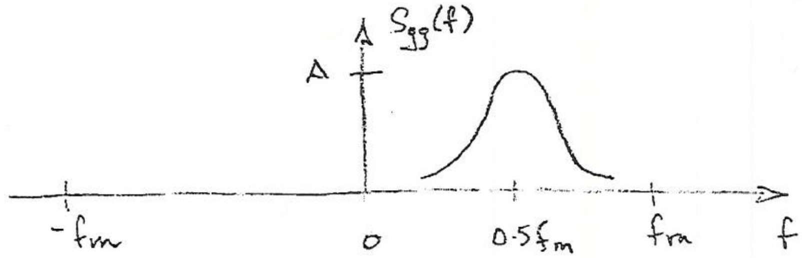
4/ 2.41 a)



b)

$$\phi_{gg}(\tau) = \mathcal{F}^{-1}\{S_{gg}(f)\}$$

From tables provided

$$e^{-\alpha\tau^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\pi^2 f^2}{\alpha}}$$

$$e^{-\alpha\tau^2} e^{j2\pi f_1\tau} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\pi^2}{\alpha}(f-f_1)^2}$$

We have,

$$\frac{\alpha^2}{\pi^2} = 2f_2^2 \Rightarrow \alpha = 2\pi^2 f_2^2$$

$$\sqrt{\frac{\pi}{\alpha}} = \sqrt{\frac{\pi}{2\pi^2 f_2^2}} = \sqrt{\frac{1}{2\pi f_2^2}}$$

Hence,

$$e^{-2\pi^2 f_2^2 \tau^2} e^{j2\pi f_1 \tau} \longleftrightarrow \sqrt{\frac{1}{2\pi f_2^2}}\, e^{-\frac{(f-f_1)^2}{2f_2^2}}$$

$$A\sqrt{2\pi}\, f_2\, e^{-2\pi^2 f_2^2 \tau^2} e^{j2\pi f_1 \tau} \longleftrightarrow A e^{-\frac{(f-f_1)^2}{2f_2^2}}$$

Using $f_1 = 0.5 f_m$, $f_2 = 0.1 f_m$

$$\phi_{gg}(\tau) = 0.1 A\sqrt{2\pi}\, f_m\, e^{-0.02\pi^2 f_m^2 \tau^2} e^{j\pi f_m \tau}$$

$$= 0.1 A\sqrt{2\pi}\, f_m\, e^{-0.02\pi^2 f_m^2 \tau^2} \times \left(\cos(\pi f_m \tau) + j \sin(\pi f_m \tau) \right)$$

c) We want

$$\phi_{g_I g_Q}(\tau) = 0$$

This occurs when $\sin(\pi f_m \tau) = 0$

or when

$$\pi f_m \tau = k\pi$$

$$\tau = \frac{k}{f_m}, \quad k = \pm 1, \pm 2, \ldots$$

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```
clear all;
fmt=0.1;
t=0:1:400;
M=8;
N=(2*M+1)*2;
alpha=0;
theta_0=pi/6;
wm=2*pi*fmt;
beta=pi*(1:M)/M;
i=1;
for t0=t,
    coswn(i,:)=cos(wm*t0*cos(2*pi*(1:M)/N));
    Gi(i)=sum(2*cos(beta).*coswn(i,:));
    Gq(i)=sum(2*sin(beta).*coswn(i,:));
    i=i+1;
end;
Gi=Gi+sqrt(2)*cos(alpha)*cos(wm*t);
Gq=Gq+sqrt(2)*sin(alpha)*cos(wm*t);
Gi=2*Gi/sqrt(2*(M+1));
Gq=2*Gq/sqrt(2*(M+1));
Gi=Gi+cos(wm*t*cos(theta_0));
Gq=Gq+sin(wm*t*cos(theta_0));
z=sqrt(Gi.*Gi+Gq.*Gq);
phase=atan2(Gq,Gi)+pi;
zdb=10*log10(z);
subplot(2,1,1), plot(zdb), title('Rayleigh faded envelope'),
xlabel('time t/T'), ylabel ('Faded Envelope (dB)'),
axis([0 400 -10 7]);
subplot(2,1,2), plot(phase), title('Rayleigh faded Envelope'),
xlabel('time t/T'), ylabel ('Faded phase'),
axis([0 400 0 2*pi]);
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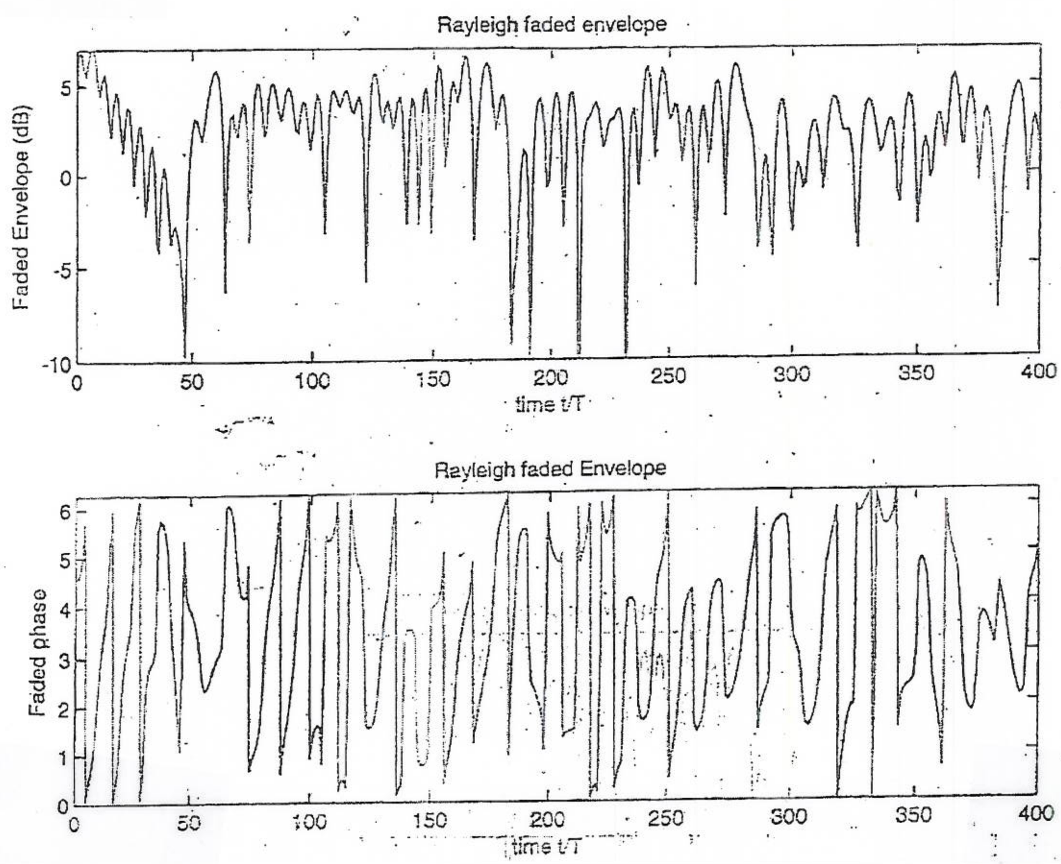

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6/ 2.51

a) $$\phi_{\Omega\Omega}(n) = E[\Omega_K \Omega_{K+n}]$$
$$= E[\Omega_K (\xi \Omega_{K+n-1} + (1-\xi) V_{K+n-1})]$$
$$= \xi E[\Omega_K \Omega_{K+n-1}] + (1-\xi) E[\Omega_K V_{K+n-1}]$$
$$= \xi \phi_{\Omega\Omega}(n-1)$$
$$= \xi^n \phi_{\Omega\Omega}(0)$$

However,

$$\phi_{\Omega\Omega}(0) = E[\Omega_K^2]$$
$$= E[(\xi \Omega_{K-1} + (1-\xi) V_{K-1})^2]$$
$$= E[\xi^2 \Omega_{K-1}^2 + 2\xi(1-\xi)\Omega_{K-1} V_{K-1} + (1-\xi)^2 V_{K-1}^2]$$
$$= \xi^2 E[\Omega_{K-1}^2] + 2\xi(1-\xi) E[\Omega_{K-1}] E[V_{K-1}] + (1-\xi)^2 E[V_{K-1}^2]$$

= cont'd.

$$= \xi^2 \phi_{\Omega\Omega}(0) + (1-\xi)^2 \tilde{\sigma}^2$$

$$\Rightarrow \quad \phi_{\Omega\Omega}(0) = \frac{(1-\xi)^2}{1-\xi^2} \tilde{\sigma}^2$$

$$= \frac{(1-\xi)^2 \tilde{\sigma}^2}{(1-\xi)(1+\xi)}$$

$$= \frac{(1-\xi)}{(1+\xi)} \tilde{\sigma}^2$$

$$\therefore \quad \phi_{\Omega\Omega}(n) = \frac{(1-\xi)}{(1+\xi)} \xi^n \tilde{\sigma}^2$$

Since $\phi_{\Omega\Omega}(n)$ is even, we have

$$\phi_{\Omega\Omega}(n) = \frac{(1-\xi)}{(1+\xi)} \xi^{|n|} \tilde{\sigma}^2$$

b) $$E[\Omega_{k+1}] = \xi E[\Omega_k] + (1-\xi) \cancelto{0}{E[V_k]}$$

$$\Rightarrow \quad E[\Omega_k] = 0$$

$$E[\Omega_{k+1}^2] = \frac{(1-\xi)}{(1+\xi)} \tilde{\sigma}^2$$