Lecture 5

Envelope Distribution
Invoking the Central Limit Theorem

• Consider the transmission of an unmodulated carrier, \( \tilde{s}(t) = 1 \), such that \( s(t) = \cos(2\pi f_c t) \).

• For flat fading channels, the received band-pass signal has the quadrature representation

\[
\begin{align*}
    r(t) &= g_I(t) \cos 2\pi f_c t - g_Q(t) \sin 2\pi f_c t \\
    \text{where}
    
    g_I(t) &= \sum_{n=1}^{N} C_n \cos \phi_n(t) \\
    g_Q(t) &= \sum_{n=1}^{N} C_n \sin \phi_n(t)
\end{align*}
\]

where \( \phi_n(t) = 2\pi \left\{ f_{D,n} t - (f_c + f_{D,n})\tau_n \right\} \).

• It is safe to assume that the phases \( \phi_n(t) \) are statistically independent random variables at any time \( t \), since the path delays \( \tau_n \) are all independent due to the random placement of scatterers. Furthermore, the phases \( \phi_n(t) \) can be treated as being uniformly distributed over the interval \([−\pi, \pi)\), since \( f_c \tau_n \gg 1 \).

• In the limit \( N \to \infty \), the central limit theorem can be invoked and \( g_I(t) \) and \( g_Q(t) \) can be treated as “Gaussian random processes,” i.e., at any time \( t_1 \), \( g_I(t_1) \) and \( g_Q(t_1) \) are Gaussian random variables.

• The “complex envelope” is

\[
g(t) = g_I(t) + jg_Q(t)
\]
• For some types of scattering environments, $g_I(t)$ and $g_Q(t)$ at any time $t_1$ are independent identically distributed Gaussian random variables with zero mean and variance $b_0 = \mathbb{E}[g_I^2(t_1)] = \mathbb{E}[g_Q^2(t_1)]$. This typically occurs in a rich scattering environment where there is no line-of-sight or strong specular component in the received signal (i.e., there is no dominant $C_n$) and isotropic antennas are used. Under such conditions, the channel exhibits “Rayleigh fading.”

• The envelope of the received signal $\alpha = |g(t_1)| = \sqrt{g_I^2(t_1) + g_Q^2(t_1)}$ is “Rayleigh” distributed at any time $t_1$, i.e.,

$$p_\alpha(x) = \frac{x}{b_0} \exp \left\{ -\frac{x^2}{2b_0} \right\} = \frac{2x}{\Omega_p} \exp \left\{ -\frac{x^2}{\Omega_p} \right\} \quad x \geq 0 ,$$

where $\Omega_p = \mathbb{E}[\alpha^2] = \mathbb{E}[g_I^2(t_1)] + \mathbb{E}[g_Q^2(t_1)] = 2b_0$ is the “average envelope power.”

• The squared-envelope $\alpha^2$ at any time $t_1$ has the exponential distribution

$$p_\alpha^2(x) = \frac{1}{\Omega_p} \exp \left\{ -\frac{x}{\Omega_p} \right\}$$
A line-of-sight (LoS) or specular (strong reflected) component arrives at angle $\theta_0$. 
For scattering environments that have a specular or LoS component, $g_I(t)$ and $g_Q(t)$ are Gaussian random processes with non-zero means $m_I(t)$ and $m_Q(t)$, respectively.

If we again assume that $g_I(t_1)$ and $g_Q(t_1)$ at any time $t_1$ are independent random variables with variance $b_0 = E[(g_I(t_1) - m_I(t_1))^2] = E[(g_Q(t_1) - m_Q(t_1))^2]$, then the magnitude of the received complex envelope $\alpha = |g(t_1)|$ at any time $t_1$ has a Rice distribution.

With Aulin’s Ricean fading model

\[
m_I(t) = E[g_I(t)] = s \cdot \cos(2\pi f_m \cos \theta_0 t + \phi_0)
\]

\[
m_Q(t) = E[g_Q(t)] = s \cdot \sin(2\pi f_m \cos \theta_0 t + \phi_0)
\]

where $f_m \cos \theta_0$ and $\phi_0$ are the Doppler shift and random phase offset associated with the LoS or specular component, respectively.

The envelope $\alpha(t) = |g(t)| = \sqrt{g_I^2(t) + g_Q^2(t)}$ is “Rice” distributed

\[
p_\alpha(x) = \frac{x}{b_0} \exp \left\{ -\frac{x^2 + s^2}{2b_0} \right\} I_0 \left( \frac{xs}{b_0} \right), \quad x \geq 0
\]

- $s^2 = m_I(t)^2 + m_Q(t)^2$ is the specular power.
- $2b_0$ is the scatter power.
- The “Rice factor,” $K = s^2/2b_0$, is the ratio of the power in the specular and scatter components.
• The average envelope power is $E[\alpha^2] = \Omega_p = s^2 + 2b_0$ and

$$s^2 = \frac{K\Omega_p}{K+1}, \quad 2b_0 = \frac{\Omega_p}{K+1}$$

Hence,

$$p_\alpha(x) = \frac{2x(K+1)}{\Omega_p} \exp\left\{-K - \frac{(K+1)x^2}{\Omega_p}\right\} I_o\left(2x\sqrt{\frac{K(K+1)}{\Omega_p}}\right), \quad x \geq 0$$

• The squared-envelope $\alpha^2(t)$ has non-central chi-square distribution with two degrees of freedom

$$p_{\alpha^2}(x) = \frac{(K+1)}{\Omega_p} \exp\left\{-K - \frac{(K+1)x}{\Omega_p}\right\} I_o\left(2\sqrt{\frac{K(K+1)x}{\Omega_p}}\right), \quad x \geq 0$$
The Rice distribution for several values of $K$ with $\Omega_p = 1$. 
Nakagami Fading

- Nakagami fading describes the magnitude of the received complex envelope by the distribution

\[
p_\alpha(x) = \frac{2m^m x^{2m-1}}{\Gamma(m) \Omega_p^m} \exp\left\{-\frac{m x^2}{\Omega_p}\right\} \quad m \geq \frac{1}{2}
\]

- When \( m = 1 \), the Nakagami distribution becomes the Rayleigh distribution, when \( m = 1/2 \) it becomes a one-sided Gaussian distribution, and when \( m \to \infty \) the distribution approaches an impulse (no fading).

- The Rice distribution can be closely approximated by using the following relation between the Rice factor \( K \) and the Nakagami shape factor \( m \)

\[
K \approx \frac{\sqrt{m^2 - m}}{m - \sqrt{m^2 - m}} \quad m > 1
\]

\[
m \approx \frac{(K + 1)^2}{(2K + 1)}.
\]

- The squared-envelope has the Gamma distribution

\[
p_{\alpha^2}(x) = \left(\frac{m}{2\Omega_p}\right)^m \frac{x^{m-1}}{\Gamma(m)} \exp\left\{-\frac{mx}{2\Omega_p}\right\}.
\]
The Nakagami pdf for several values of $m$ with $\Omega_p = 1$. 