level Crossing Rate, Average Fade Duration, Zero Crossing Rate
The level crossing rate (LCR) is the rate at which the received envelope crosses a specified level in the positive (or negative) going direction.

- The LCR can be used to estimate velocity.

The average fade duration (AFD) is the average length of time that the envelope remains below a specified level.

- The AFD is impacts the outage probability and quality of service.

Both the LCR and AFD are second-order statistics that depend on the mobile station velocity, as well as the scattering environment.

The LCR and AFD have been derived by Rice in the context of a sinusoid in narrowband Gaussian noise.
Rayleigh faded envelope with 2-D isotropic scattering.
Level Crossing Rate

• Obtaining the level crossing rate requires the joint pdf, $p(\alpha, \dot{\alpha})$, of the envelope level $\alpha = |g|$ and the envelope slope $\dot{\alpha}$.

• In terms of $p(\alpha, \dot{\alpha})$, the expected amount of time spent in the interval $(R, R + d\alpha)$ for a given envelope slope $\dot{\alpha}$ and time duration $dt$ is

$$p(R, \dot{\alpha})d\alpha d\dot{\alpha}dt$$

• The time required to cross the level $\alpha$ once for a given envelope slope $\dot{\alpha}$, in the interval $(R, R + d\alpha)$ is

$$d\alpha/\dot{\alpha}$$

• The ratio of the above two quantities is the expected number of crossings of the envelope $\alpha$ within the interval $(R, R + d\alpha)$ for a given envelope slope $\dot{\alpha}$ and time duration $dt$, i.e.,

$$\dot{\alpha}p(R, \dot{\alpha})d\dot{\alpha}dt$$
The expected number of crossings of the envelope level \( R \) for a given envelope slope \( \dot{\alpha} \) in a time interval of duration \( T \) is
\[
\int_{0}^{T} \dot{\alpha}p(R, \dot{\alpha})d\dot{\alpha}dt = \dot{\alpha}p(R, \dot{\alpha})d\dot{\alpha}T
\]

The expected number of crossings of the envelope level \( R \) with a positive slope is
\[
N_{R} = T \int_{0}^{\infty} \dot{\alpha}p(R, \dot{\alpha})d\dot{\alpha}
\]

Finally, the expected number of crossings of the envelope level \( R \) per second, or the level crossing rate, is
\[
L_{R} = \int_{0}^{\infty} \dot{\alpha}p(R, \dot{\alpha})d\dot{\alpha}
\]

This is a general result that applies to any random process.

Rice has derived the joint pdf \( p(\alpha, \dot{\alpha}) \) for a sine wave plus Gaussian noise. A Rician fading channel consists of LoS or specular (sine wave) component plus a scatter (Gaussian noise) component. For the case of a Rician fading channel,
\[
p(\alpha, \dot{\alpha}) = \frac{\alpha(2\pi)^{-3/2}}{\sqrt{Bb_0}} \int_{-\pi}^{\pi} d\theta
\times \exp \left\{-\frac{1}{2Bb_0} \left[ B \left( \alpha^2 - 2\alpha s \cos \theta + s^2 \right) + (b_0 \dot{\alpha} + b_1 s \sin \theta)^2 \right] \right\}
\]
where \( s \) is the non-centrality parameter in the Rice distribution, and \( B = b_0b_2 - b_1^2 \), where \( b_0, b_1, \) and \( b_2 \) are constants that depend on the scattering environment.
Suppose that the specular or LoS component of the complex envelope $g(t)$ has a Doppler frequency equal $f_q = f_m \cos \theta_0$, where $0 \leq |f_q| \leq f_m$. Then

$$b_n = (2\pi)^n \int_{-f_m}^{f_m} S_{gg}^c(f)(f - f_q)^n df$$

$$= (2\pi)^n b_0 \int_0^{2\pi} \hat{p}(\theta) (f_m \cos \theta - f_q)^n d\theta$$

where $\hat{p}(\theta)$ is the azimuth distribution (pdf) of the scatter component and $S_{gg}^c(f)$ is the corresponding continuous portion of the Doppler power spectrum.

Note that $S_{gg}^c(f)$ is given by the Fourier transform of $\phi_{gg}^c(\tau) = \phi_{gIgI}^c(\tau) + j \phi_{gIgQ}^c(\tau)$ where

$$\phi_{gIgI}^c(\tau) = \frac{\Omega_p}{2} \int_0^{2\pi} \cos(2\pi f_m \tau \cos \theta) \hat{p}(\theta) d\theta$$

$$\phi_{gIgQ}^c(\tau) = \frac{\Omega_p}{2} \int_0^{2\pi} \sin(2\pi f_m \tau \cos \theta) \hat{p}(\theta) d\theta$$
• In some special cases, the psd $S^c_{gg}(f)$ is symmetrical about the frequency $f_q = f_m \cos \theta_0$. This condition occurs, for example, when $f_q = 0 \ (\theta_0 = 90^\circ)$ and $\hat{p}(\theta) = 1/(2\pi), -\pi \leq \theta \leq \pi$.

• In this case, $b_n = 0$ for all odd values of $n$ (and in particular $b_1 = 0$) so that the joint pdf $p(\alpha, \dot{\alpha})$ reduces to the convenient product form

$$p(\alpha, \dot{\alpha}) = \sqrt{\frac{1}{2\pi b_2}} \exp \left\{ -\frac{\dot{\alpha}^2}{2b_2} \right\} \cdot \frac{\alpha}{b_0} \exp \left\{ -\frac{(\alpha^2 + s^2)}{2b_0} \right\} I_0 \left( \frac{\alpha s}{b_0} \right)$$

$$= p(\dot{\alpha}) \cdot p(\alpha).$$

• Since $p(\alpha, \dot{\alpha}) = p(\dot{\alpha}) \cdot p(\alpha)$, it follows that $\alpha$ and $\dot{\alpha}$ are independent.
• When $f_q = 0$ and $\hat{p}(\theta) = 1/(2\pi)$, a closed form expression can be obtained for the envelope level crossing rate.

• We have that

$$b_n = \begin{cases} 
  b_0(2\pi f_m)^{n1 \cdot 3 \cdot 5 \cdots (n-1)} & n \text{ even} \\
  0 & n \text{ odd}
\end{cases} \cdot \frac{2 \cdot 4 \cdot 6 \cdots n}{2^{n/2}}$$

• Therefore, $b_1 = 0$ and $b_2 = b_0(2\pi f_m)^2/2$, and

$$L(R) = \sqrt{2\pi(K+1)} f_m \rho e^{-K-(K+1)\rho^2} I_0 \left(2\rho \sqrt{K(K+1)}\right)$$

where

$$\rho = \frac{R}{\sqrt{\Omega_p}} = \frac{R}{R_{\text{rms}}}$$

and $R_{\text{rms}} \triangleq \sqrt{\text{E}[\alpha^2]}$ is the rms envelope level.

• Under the further condition that $K = 0$ (Rayleigh fading)

$$L(R) = \sqrt{2\pi} f_m \rho e^{-\rho^2}$$

• Notice that the level crossing rate is directly proportional to the maximum Doppler frequency $f_m$ and, hence, the MS speed $v = f_m/\lambda_c$. 
Normalized level crossing rate for Rician fading. A specular component arrives with angle $\theta_0 = 90^\circ$ and there is 2-D isotropic scattering of the scatter component.
No known expression exists for the duration of fades; an open problem! Therefore, we consider the “average fade duration”.

Consider a very long time interval of length $T$, and let $t_i$ be the duration of the $i$th fade below the level $R$.

The probability that the received envelope is less than $R$ is

$$\Pr[r \leq R] = \frac{1}{T} \sum_i t_i$$

The average fade duration is equal to

$$\bar{t} = \frac{1}{TL(R)} \sum_i t_i = \frac{\Pr[r \leq R]}{L(R)}$$
• If the envelope is Rician distributed, then

\[ \Pr[r \leq R] = \int_0^R p(r)dr = 1 - Q\left(\sqrt{2K}, \sqrt{2(K + 1)\rho^2}\right) \]

where \( Q(a, b) \) is the Marcum Q function.

• if we again assume that \( f_q = 0 \) and \( \hat{p}(\theta) = 1/(2\pi) \), we have

\[ \bar{t} = \frac{1 - Q\left(\sqrt{2K}, \sqrt{2(K + 1)\rho^2}\right)}{\sqrt{2\pi(K + 1)f_m\rho e^{-K - (K + 1)\rho^2} I_0\left(2\rho\sqrt{K(K + 1)}\right)}} \]

• If we further assume that \( K = 0 \) (Rayleigh fading), then

\[ P(\alpha \leq R) = \int_0^R p(\alpha)d\alpha = 1 - e^{-\rho^2} \]

and

\[ \bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m\sqrt{2\pi}} \]
Normalized average fade duration with Ricean fading.
Zero Crossing Rate

- Recall that received complex envelope $g(t) = g_I(t) + g_Q(t)$ is a complex Gaussian random process. If the channel is characterized by a specular or LoS component, then $g_I(t)$ and $g_Q(t)$ have mean values $m_I(t)$ and $m_Q(t)$, respectively. Here we are interested in the “zero crossing rate” of the zero-mean Gaussian random processes $\hat{g}_I(t) = g_I(t) - m_I(t)$ and $\hat{g}_Q(t) = g_Q(t) - m_Q(t)$.

- Rice has derived this zero crossing rate as

$$L_Z = \frac{1}{\pi} \sqrt{\frac{b_2}{b_0}}.$$  

- When the scatter component has the azimuth distribution $\hat{p}(\theta) = 1/(2\pi)$, $-\pi \leq \theta \leq \pi$, the zero crossing rate is

$$L_Z = \sqrt{2} f_m.$$  

- Similar to the level crossing rate, the zero crossing rate is directly proportional to the maximum Doppler frequency $f_m = v/\lambda_c$. 

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