EE4061
Communication Systems

Week 11

Intersymbol Interference

Nyquist Pulse Shaping
Intersymbol Interference (ISI)

\[ \sum_n a_n \delta(t-nT) \]

\[ \{a_k\} \]

\[ a_k \in \{-1,+1\} \]

Tx filter \[ g(t) \]

channel \[ c(t) \]

\[ x(t) \]

Rx filter \[ h(t) \]

\[ y(t) \]

\[ w(t) \]

AWGN

An ideal channel \[ c(t) \] only scales and time shifts the signal \[ g(t) \], but otherwise leaves it undistorted, i.e. for an ideal channel

\[ c(t) = \alpha \delta(t - t_o) \]

\[ g(t) * c(t) = \alpha g(t - t_o) \]

\[ C(f) = \alpha e^{-j2\pi ft_o} \]
Amplitude and phase response for an ideal channel.

For a more general, non-ideal, channel, let

\[ p(t) = g(t) * c(t) * h(t) \]
\[ \uparrow \]
\[ P(f) = G(f)C(f)H(f) \]

Then \( y(t) = \sum_n a_n p(t - nT) + n(t) \), where \( n(t) = w(t) * h(t) \)
ISI

\[ y_i = y(iT) = \sum_n a_n p((i - n)T) + n(iT) \]
\[ = \sum_n a_n p_{i-n} + n_i \]

where \( p_{i-n} = p((i - n)T) \), \( n_i = nT \)

\[ y_i = a_i p_0 + \sum_{n \neq i} a_n p_{i-n} + n_i \]

\( a_i p_0 \) – desired term,
\( \sum_{n \neq i} a_n p_{i-n} \) – ISI
\( n_i \) – noise

In the absence of ISI and noise, \( y_i = a_i p_0 \). Any pulse \( p(t) \), such that the sampled pulse satisfies the condition

\[ p_i = p(iT) = \begin{cases} p_0 & i = 0 \\ 0 & i \neq 0 \end{cases} = p_0 \delta_{i0} \]

yields zero ISI

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What overall pulse shapes \( p(t) \), \( p(t) = g(t) * c(t) * h(t) \), will yield zero ISI?

Suppose \( P(f) = \frac{1}{2W} \text{rect} \left( \frac{f}{2W} \right) \), where \( W = 1/2T = R/2 \), \( T \) is the baud duration, \( R \) is the baud rate.

\[
p(t) = \text{sinc}(2Wt) = \text{sinc}(t/T), \quad T = 1/2W.
\]

Note that
\[
p_i = p(iT) = \begin{cases} 
1 & i = 0 \\
0 & i \neq 0 
\end{cases} = \delta_{i0}
\]

This pulse results in zero ISI. Note that \( p(t) \) is noncausal.
Problems:
1. $P(f) = \frac{1}{2W} \text{rect} \left( \frac{f}{2W} \right)$ is an ideal low pass filter that is not realizable.
2. $p(t)$ decays slowly with time. It decreases with $1/|t|$ for large $t$. Therefore, it is very sensitive to sampler phase, i.e., a small error in the sampler timing phase can lead to significant ISI.

We desire a pulse $p(t)$ that is realizable and has tails that decay quickly in time.
Problems with ‘sinc’ pulse

\[ y(t) = \sum_n a_n p(t - nT) \]
\[ y(\Delta t) = \sum_n a_n p(\Delta t - nT) \]
\[
= \sum_n a_n \frac{\sin[\pi(\Delta t - nT)/T]}{\pi(\Delta t - nT)/T}
\]
\[
= \sum_n \frac{\sin(\pi \Delta t/T) \cos(\pi n) - \cos(\pi \Delta t/T) \sin(\pi n)}{\pi \Delta t/T - n\pi}
\]
\[
= \sum_n a_n \frac{(-1)^n \sin(\pi \Delta t/T)}{\pi \Delta t/T - n\pi}
\]
\[
= a_0 \text{sinc}(\Delta t/T) + \frac{\sin(\pi \Delta t/T)}{\pi} \sum_{n \neq 0} \frac{a_n (-1)^n}{\Delta t/T - n}
\]

Last term is not absolutely summable.

We have seen \( y_i = y(iT) = a_i p_0 + \sum_{n \neq i} a_n p_{i-n} + n_i \)
where \( p_k = p(kT), n_i = n(iT) \).
Matched Filtering and Pulse Shaping

- To maximize the signal-to-noise ratio at the output of the receiver filter \( h(t) \), in theory we match the receiver filter to the received pulse \( \hat{g}(t) = g(t) \ast c(t) \), i.e., \( h(t) = \hat{g}(T - t) \). However, if \( c(t) \) is unknown, then so is \( h(t) \).

- **Practical Solution:** Choose \( h(t) \) matched to the transmitted pulse \( g(t) \), i.e., choose \( h(t) = g(T - t) \), over-sample by a factor of 2, and process 2 samples per baud interval.
  
  - This is optimal, similar to the case when \( c(t) \) is known, but the proof is beyond the scope of this course.
Matched Filtering and Pulse Shaping

- To design the transmit and receiver filters, we will assume an ideal channel $c(t) = \delta(t)$, so that the overall pulse (ignoring time delay) is

$$p(t) = g(t) * h(t) = g(t) * g(-t)$$

- Taking the Fourier transform of both sides

$$P(f) = G(f)G^*(f) = |G(f)|^2$$

- Hence

$$|G(f)| = \sqrt{|P(f)|}$$

- For many practical pulses, $g(t)$, we will also see that $g(t) = g(-t)$, i.e., the pulse is even in $t$, so that $h(t) = g(t)$. This means that the transmit and receive matched filters are identical filters.
Conditions for ISI free transmission

The condition for ISI-free transmission is

\[ p_k = \delta_{k0} p_0 = \begin{cases} p_0 & k = 0 \\ 0 & k \neq 0 \end{cases} \]

That is, \( p(t) \) must have equally spaced zero crossings, separated by \( T \) seconds.

**Theorem:** The pulse \( p(t) \) satisfies \( p_k = \delta_{k0} p_0 \) iff

\[ P_\Sigma(f) \triangleq \frac{1}{T} \sum_{n=-\infty}^{\infty} P(f + n/T) = p_0 \]

That is the folded spectrum \( P_\Sigma(f) \) is flat.

[Diagram showing pulse and folded spectrum]

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ISI free transmission

Proof:

\[
p_k = \int_{-\infty}^{\infty} P(f) e^{j2\pi kT} df
\]

\[
= \sum_{n=-\infty}^{\infty} \int_{(2n-1)/2T}^{(2n+1)/2T} P(f) e^{j2\pi f kT} df \quad f' = f - n/T
\]

\[
= \sum_{n=-\infty}^{\infty} \int_{-1/2T}^{1/2T} P(f' + n/T) e^{j2\pi k(f' + n/T)T} df'
\]

\[
= \int_{-1/2T}^{1/2T} e^{j2\pi f' kT} \left[ \sum_{n=-\infty}^{n=\infty} P(f' + n/T) \right] df'
\]

(1)

To prove sufficiency, we assume that \( \sum_{n=-\infty}^{\infty} P(f' + n/T) = p_0 T \) is true. Then,

\[
p_k = p_0 T \int_{-1/2T}^{1/2T} e^{j2\pi f' kT} df' = \frac{\sin \pi k}{\pi k} p_0 = \delta_{k0} p_k
\]

To prove necessity, we have from (1)

\[
p_k = T \int_{-1/2T}^{1/2T} P_\Sigma(f') e^{j2\pi f' kT} df'
\]
Nyquist Pulse

Hence, $p_k$ and $P_\Sigma(f)$ are a Fourier series pair, i.e.,

$$P_\Sigma(f) = \sum_{k=-\infty}^{\infty} p_k e^{-j2\pi f k T}$$

If $p_k = p_0 \delta_{k0}$ is assumed true, then from the above equation $P_\Sigma(f) = p_0$.

- **Nyquist Pulse Shaping:** A pulse $p(t)$ that yields zero-ISI is one having a folded spectrum that is flat.
  - The pulse $p(t)$ can be generated by choosing $P(f)$ as shown on the following slide.
Nyquist Pulse Shaping

Note \( P(f) = P_N(f) + P_{od}(f) \).
\( P_{od}(f) \) can be any function that has skew symmetry about \( f = W = 1/2T \).
Nyquist Pulse

Note that $P_\Sigma(f)$ is flat under this condition.

Example: Raised Cosine

\[ 2WP_{od}(f) = \begin{cases} 
-\frac{1}{2} - \frac{1}{2} \sin \frac{\pi(|f|-W)}{2f_x} & W - f_x \leq |f| \leq W \\
\frac{1}{2} - \frac{1}{2} \sin \frac{\pi(|f|-|W|)}{2f_x} & W \leq |f| \leq W + f_x 
\end{cases} \]

$f_x = \text{bandwidth expansion, } \frac{f_x}{W} \times 100 = \text{excess bandwidth (\%)}, \alpha = \frac{f_x}{W} = \text{roll off factor}$

\[ 2WP(f) = \begin{cases} 
1 & 0 \leq |f| \leq W - f_x \\
\frac{1}{2} \left[ 1 - \sin \frac{\pi(|f|-W)}{2f_x} \right] & W - f_x \leq |f| \leq W + f_x \\
0 & |f| \geq W + f_x 
\end{cases} \]

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Raised Cosine Pulse

\[ 2WP(f) \]

\[ \alpha = 1 \]

looks like a raised cosine

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Raised Cosine Impulse Response

Impulse response - Since $P(f)$ is even, the inverse cosine transform yields

\[
p(t) = 2 \int_0^{W+f_x} P(f) \cos 2\pi f t df
\]

\[
= 2 \cdot \frac{1}{2W} \int_0^{W-f_x} \cos 2\pi f t df + 2 \cdot \frac{1}{2W} \int_{W-f_x}^{W+f_x} \frac{1}{2} \left[ 1 - \sin \frac{\pi |f| - W}{2f_x} \right] \cos 2\pi f t df
\]

\[
= \frac{\sin 2\pi W t}{2\pi W t} \frac{\cos 2\pi f_x t}{1 - (4f_x t)^2}
\]
Square Root Raised Cosine Pulse

- To implement a matched filter, we split the overall pulse $P(f)$ between the transmit and receive filters, i.e., $p(t) = g(t) * g(-t)$.
- However, $P(f) = G(f)G^*(f) = |G(f)|^2$, so that $|G(f)| = \sqrt{P(f)}$.
- With square-root raised cosine pulse shaping

$$\sqrt{2W}|G(f)| = \begin{cases} 
1 & 0 \leq |f| \leq wW - f_x \\
\sqrt{2} \left[1 - \sin \frac{\pi(|f| - W)}{2f_x}\right] & W - f_x \leq |f| \leq W + f_x \\
0 & |f| \geq W + f_x
\end{cases}$$

- The impulse response is

$$g(t) = \begin{cases} 
1 - \beta + 4\beta/\pi & t = 0 \\
(\beta/\sqrt{2}) ((1 + 2/\pi) \sin(\pi/4\beta) + (1 - 2/\pi) \cos(\pi/4\beta)) & t = \pm T/4\beta \\
\frac{4\beta(t/T) \cos(1+\beta)t/T + \sin(1-\beta)t/T}{\pi(t/T)(1-(4\beta t/T)^2)} & \text{elsewhere}
\end{cases}$$

where $\alpha = f_x/W$. 
Raised cosine and root raised cosine pulses with roll-off factor $\alpha = 0.5$. The pulses are truncated to length $6T$ and time shifted by $3T$ to yield causal pulses.
M-ary QAM

- Quadrature Amplitude Modulation (QAM), the transmitted waveform in each baud interval takes on one of the following $M$ waveforms

$$s_m(t) = \sqrt{\frac{2E_0}{T}} g(t) \left( a_m^c \cos(2\pi f_c t) - a_m^s \sin(2\pi f_c t) \right)$$

where

$$a_m^{c,s} \in \{\pm 1, \pm 3, \pm 5, \pm (M - 1)\}$$

and $2E_0$ is the energy of the signal with the lowest amplitude, i.e., when $a_m^c, a_m^s = \pm 1$.

- You have seen this before for the case $g(t) = u_T(t)$; however, practical systems will use the root-raised cosine pulse for $g(t)$. Note that we use the normalization,

$$E_g = \int_{-\infty}^{\infty} g^2(t) dt = 1.$$
Eye diagram when $P(f)$ is an ideal low pass filter.
Eye diagram when $P(f)$ is a raised cosine filter with $\beta = 0.35$. 