EE4601
Communication Systems

Week 6
Orthogonal Expansions
Basic Problem

Problem:

Suppose that we have a set of $M$ finite energy signals $S = \{s_1(t), s_2(t), \ldots, s_M(t)\}$, where each signal has a duration $T$ seconds.

Every $T$ seconds one of the waveforms from the set $S$ is selected for transmission over an AWGN channel. The transmitted waveform is

$$x(t) = \sum_n s_n(t - nT)$$

The received noise corrupted waveform is

$$r(t) = \sum_n s_n(t - nT) + n(t)$$

By observing $r(t)$ we wish to determine the time sequence of waveforms $\{s_n(t)\}$ that was transmitted. That is, in each $T$ second interval, we must determine which $s_i(t) \in S$ was transmitted.
Orthogonal Expansions

Consider a real valued signal \( s(t) \) with finite energy \( E_s \),

\[
E_s = \int_{-\infty}^{\infty} s^2(t) dt
\]

Suppose there exists a set of orthornormal functions \( \{f_n(t)\} \), \( n = 1, \ldots, N \). By orthornormal we mean

\[
\int_{-\infty}^{\infty} f_n(t) f_k(t) dt = \delta_{kn}
\]

\[
\delta_{kn} = \begin{cases}
1 & , \ k = n \\
0 & , \ k \neq n
\end{cases}
\]

We now approximate \( s(t) \) as the weighted linear sum

\[
\hat{s}(t) = \sum_{k=1}^{N} s_k f_k(t)
\]

and wish to determine the \( s_k \), \( k = 1, \ldots, N \) to minimize the square error

\[
\varepsilon = \int_{-\infty}^{\infty} (s(t) - \hat{s}(t))^2 dt
\]

\[
= \int_{-\infty}^{\infty} \left( s(t) - \sum_{k=1}^{N} s_k f_k(t) \right)^2 dt
\]
Orthogonal Expansions

To minimize the mean square error, we take the partial derivative with respect to each of the $s_k$ and set equal to zero, i.e., for the $n$th term we solve

$$\frac{\partial \varepsilon}{\partial s_n} = 2 \int_{-\infty}^{\infty} \left( s(t) - \sum_{k=1}^{N} s_k f_k(t) \right) f_n(t) dt = 0.$$ 

Using the orthonormal property of the basis functions, $s_n = \int_{-\infty}^{\infty} s(t) f_n(t) dt$ and

$$\varepsilon = \int_{-\infty}^{\infty} \left( s(t) - \sum_{k=1}^{N} s_k f_k(t) \right)^2 dt$$

$$= \int_{-\infty}^{\infty} s^2(t) dt - 2 \int_{-\infty}^{\infty} s(t) \sum_{k=1}^{N} s_k f_k(t) dt + \int_{-\infty}^{\infty} \sum_{k=1}^{N} s_k f_k(t) \sum_{\ell=1}^{N} s_\ell f_\ell(t) dt$$

$$= \int_{-\infty}^{\infty} s^2(t) dt - 2 \sum_{k=1}^{N} s_k \int_{-\infty}^{\infty} s(t) f_k(t) dt + \sum_{k=1}^{N} \sum_{\ell=1}^{N} s_k s_\ell \int_{-\infty}^{\infty} f_k(t) f_\ell(t) dt$$

$$= E_s - \sum_{k=1}^{N} s_k^2$$

For a complete set of basis functions $\varepsilon = 0$. 
Suppose that we have a set of finite energy real signals \( \{s_i(t)\}, i = 1, \ldots, M \). We wish to obtain a complete set of orthonormal basis functions for the signal set. This can be done in 2 steps.

**Step 1:** Determine if the set of waveforms is linearly independent. If they are linearly dependent, then there exists a set of coefficients \( a_1, a_2, \ldots, a_M \), not all zero, such that

\[
a_1 s_1(t) + a_2 s_2(t) + \cdots + a_M s_M(t) = 0.
\]

Suppose, without loss of generality, that \( a_M \neq 0 \). If \( a_M = 0 \), then the signal set can be permuted so that \( a_M \neq 0 \). Then

\[
s_M(t) = -\left( \frac{a_1}{a_M} s_1(t) + \frac{a_2}{a_M} s_2(t) + \cdots + \frac{a_{M-1}}{a_M} s_{M-1}(t) \right).
\]

Next consider the reduced signal set \( \{s_i(t)\}_{i=1}^{M-1} \). If this set of waveforms is linearly dependent, then there exists another set of co-efficients \( \{b_i\}_{i=1}^{M-1} \), not all zero, such that

\[
b_1 s_1(t) + b_2 s_2(t) + \cdots + b_{M-1} s_{M-1}(t) = 0.
\]
Gram-Schmidt Orthonormalization

We continue until a set \( \{ s_i(t) \}_{i=1}^{N} \) of linearly independent waveforms is obtained. Note that \( N \leq M \) with equality if and only if the set of waveforms \( \{ s_i(t) \}_{i=1}^{M} \) is linearly independent.

If \( N < M \), then the set of linearly independent waveforms \( \{ s_i(t) \}_{i=1}^{N} \) is not unique, but any one will do.

**Step 2:** From the set \( \{ s_i(t) \}_{i=1}^{N} \) construct the set of \( N \) orthonormal basis functions \( \{ f_i(t) \}_{i=1}^{N} \) as follows. First, let

\[
f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}
\]

where \( E_1 \) is the energy in the waveform \( s_1(t) \), given by

\[
E_1 = \int_0^T s_1^2(t)dt
\]

Then

\[
s_1(t) = \sqrt{E_1} f_1(t) = s_{11} f_1(t)
\]

where \( s_{11} = \sqrt{E_1} \).
Next, by using the waveform $s_2(t)$ we obtain

$$s_{21} = \int_0^T s_2(t)f_1(t)dt$$

along with the intermediate function

$$g_2(t) = s_2(t) - s_{21}f_1(t)$$

Note that $g_2(t)$ is orthogonal to $f_1(t)$. The second basis function is

$$f_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T (g_2(t))^2 dt}} = \frac{s_2(t) - s_{21}f_1(t)}{\sqrt{E_2 - s_{21}^2}}$$
Continuing in the above fashion, we define the $i$th intermediate function

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} f_j(t)$$

where

$$s_{ij} = \int_0^T s_i(t) f_j(t) dt$$

The set of functions

$$f_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T (g_i(t))^2}} \quad i = 1, 2, \ldots, N$$

is the required set of complete orthonormal basis functions.
Gram-Schmidt Orthonormalization

We can now write the signals as weighted linear combinations of the basis functions, i.e.,

\[ s_1(t) = s_{11}f_1(t) \]
\[ s_2(t) = s_{21}f_1(t) + s_{22}f_2(t) \]
\[ s_3(t) = s_{31}f_1(t) + s_{32}f_2(t) + f_3(t) \]
\[ \vdots = \vdots \]
\[ s_N(t) = s_{N1}f_1(t) + \cdots + s_{NN}f_N(t) \]

For the remaining signals \( s_i(t), i = N + 1, \ldots, M \), we have

\[ s_i(t) = \sum_{k=1}^{N} s_{ik} f_k(t) \]

where

\[ s_{ik} = \int_{0}^{T} s_i(t) f_k(t) dt \]
Signal Vectors

It follows that the signal set $s_i(t), i = 1, \ldots, M$ can be expressed in terms of a set of signal vectors $s_i, i = 1, \ldots, M$ in an $N$-dimensional signal space, i.e.,

\[
\begin{align*}
  s_1(t) & \leftrightarrow s_1 = (s_{11}, s_{12}, \ldots, s_{1N}) \\
  s_2(t) & \leftrightarrow s_2 = (s_{21}, s_{22}, \ldots, s_{2N}) \\
  \vdots & = \vdots \\
  s_M(t) & \leftrightarrow s_M = (s_{M1}, s_{M2}, \ldots, s_{MN})
\end{align*}
\]
Example

\begin{align*}
s_1(t) &= 1, \quad 0 \leq t < \frac{T}{3} \\
        &= 0, \quad \frac{T}{3} \leq t < T \\

s_2(t) &= 1, \quad 0 \leq t < 2\frac{T}{3} \\
        &= 0, \quad 2\frac{T}{3} \leq t < T \\

s_3(t) &= 1, \quad 0 \leq t < T \\

s_4(t) &= 1, \quad 0 \leq t < T
\end{align*}
Example

Step 1: This signal set is not linearly independent because

\[ s_4(t) = s_1(t) + s_3(t) \]

Therefore, we will use \( s_1(t) \), \( s_2(t) \), and \( s_3(t) \) to obtain the complete orthonormal set of basis functions.

Step 2:

a)

\[ E_1 = \int_0^T s_1(t) dt = T/3 \]

\[ f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \frac{\sqrt{3/T}}{T} & 0 \leq t \leq T/3 \\ 0 & \text{else} \end{cases} \]
Example

b)

\[ s_{21} = \int_0^T s_2(t) f_1(t) \, dt = \int_0^{T/3} \sqrt{3/T} \, dt = \sqrt{T/3} \]

\[ E_2 = \int_0^T s_2^2(t) \, dt = 2T/3 \]

\[ f_2(t) = \frac{s_2(t) - s_{21} f_1(t)}{\sqrt{E_2 - s_{21}^2}} \]

\[ = \begin{cases} 
\sqrt{3/T} & , \quad T/3 \leq t \leq 2T/3 \\
0 & , \quad \text{else}
\end{cases} \]

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Example

c)

\[ s_{31} = \int_{0}^{T} s_{3}(t) f_{1}(t) dt = 0 \]
\[ s_{32} = \int_{0}^{T} s_{3}(t) f_{2}(t) dt \]
\[ = \int_{T/3}^{2T/3} \sqrt{3/T} dt = \sqrt{T/3} \]

\[ g_{3}(t) = s_{3}(t) - s_{31} f_{1}(t) - s_{32} f_{2}(t) \]
\[ = \begin{cases} 1 & , 2T/3 \leq t \leq T \\ 0 & , \text{else} \end{cases} \]

\[ f_{3}(t) = \frac{g_{3}(t)}{\sqrt{\int_{0}^{T} g_{3}^{2}(t) dt}} \]
\[ = \begin{cases} \sqrt{3/T} & , 2T/3 \leq t \leq T \\ 0 & , \text{else} \end{cases} \]
Example

\begin{align*}
& \mathcal{L}_1(t) = \begin{cases} 
1 & 0 \leq t < \frac{T}{3} \\
0 & \frac{T}{3} \leq t < \frac{2T}{3} \\
1 & \frac{2T}{3} \leq t \leq T
\end{cases} \\
& \mathcal{L}_2(t) = \begin{cases} 
1 & 0 \leq t < \frac{T}{3} \\
0 & \frac{T}{3} \leq t < \frac{2T}{3} \\
1 & \frac{2T}{3} \leq t \leq T
\end{cases} \\
& \mathcal{L}_3(t) = \begin{cases} 
1 & 0 \leq t < \frac{T}{3} \\
0 & \frac{T}{3} \leq t < \frac{2T}{3} \\
1 & \frac{2T}{3} \leq t \leq T
\end{cases}
\end{align*}
Example

\[ s_1(t) \leftrightarrow s_1 = (\sqrt{T/3}, 0, 0) \]
\[ s_2(t) \leftrightarrow s_2 = (\sqrt{T/3}, \sqrt{T/3}, 0) \]
\[ s_3(t) \leftrightarrow s_3 = (0, \sqrt{T/3}, \sqrt{T/3}) \]
\[ s_4(t) \leftrightarrow s_4 = (\sqrt{T/3}, \sqrt{T/3}, \sqrt{T/3}) \]
Properties of Signal Vectors

Signal Energy:

\[ E = \int_0^T s^2(t) dt \]
\[ = \int_0^T \sum_{k=1}^N s_k f_k(t) \sum_{\ell=1}^N s_\ell f_\ell(t) \, dt \]
\[ = \sum_{k=1}^N \sum_{\ell=1}^N s_k s_\ell \int_0^T f_k(t) f_\ell(t) dt \]
\[ = \sum_{k=1}^N s_k^2 \]
\[ \triangleq ||s||^2 \]

The energy in \( s(t) \) is just the squared length of its signal vector \( s \).
Properties of Signal Vectors

Signal Correlation: The correlation or “similarity” between two signals $s_j(t)$ and $s_k(t)$ is

$$\rho_{jk} = \frac{1}{\sqrt{E_j E_k}} \int_0^T s_j(t)s_k(t)dt$$

$$= \frac{1}{\sqrt{E_j E_k}} \int_0^T \sum_{n=1}^N s_{jn} f_n(t) \sum_{m=1}^N s_{km} f_m(t) dt$$

$$= \frac{1}{\sqrt{E_j E_k}} \sum_{n=1}^N \sum_{m=1}^N s_{jn} s_{kn} \int_0^T f_n(t)f_m(t) dt$$

$$= \frac{1}{\sqrt{E_j E_k}} \sum_{n=1}^N s_{jn} s_{kn}$$

$$= \frac{s_j \cdot s_k}{||s_j|| \ ||s_k||}$$

Note that

$$\rho = \begin{cases} 0 & \text{if } s_j(t) \text{ and } s_k(t) \text{ are orthogonal} \\ \pm 1 & \text{if } s_j(t) = \pm s_k(t) \end{cases}$$
Properties of Signal Vectors

**Euclidean Distance:** The Euclidean distance between two signals $s_j(t)$ and $s_k(t)$ is

$$d_{jk} = \left\{ \int_0^T (s_j(t) - s_k(t))^2 dt \right\}^{1/2}$$

$$= \left\{ \int_0^T \left( \sum_{n=1}^N s_{jn}f_n(t) - \sum_{m=1}^N s_{km}f_m(t) \right)^2 dt \right\}^{1/2}$$

$$= \left\{ \sum_{n=1}^N (s_{jn} - s_{kn})^2 \right\}^{1/2}$$

$$= \left\{ \| s_j - s_k \|^2 \right\}^{1/2}$$

$$= \| s_j - s_k \|$$
Example

Consider the earlier example where

\[\begin{align*}
  s_1 &= (\sqrt{T/3}, 0, 0) \\
  s_2 &= (\sqrt{T/3}, \sqrt{T/3}, 0) \\
  s_3 &= (0, \sqrt{T/3}, \sqrt{T/3})
\end{align*}\]

We have \(E_1 = \|s_1\|^2 = T/3\), \(E_2 = \|s_2\|^2 = 2T/3\), and \(E_3 = \|s_3\|^2 = 2T/3\).

The correlation between \(s_2(t)\) and \(s_3(t)\) is

\[\rho_{23} = \frac{s_2 \cdot s_3}{\|s_2\| \|s_3\|} = \frac{T/3}{2T/3} = 0.5\]

The Euclidean distance between \(s_1(t)\) and \(s_3(t)\) is

\[d_{13} = \|s_1 - s_3\| = \sqrt{T/3 + T/3 + T/3} = \sqrt{T}\]