EE4601
Communication Systems

Week 3
Random Processes, Stationarity, Means, Correlations
Random Processes

A random process or stochastic process, $X(t)$, is an ensemble of $\zeta$ sample functions $\{X_1(t), X_2(t), \ldots, X_\zeta(t)\}$ together with a probability rule which assigns a probability to any meaningful event associated with the observation of these sample functions.

Suppose the sample function $X_i(t)$ corresponds to the sample point $s_i$ in the sample space $S$ and occurs with probability $P_i$.

- $\zeta$ may be finite or infinite.
- Sample functions may be defined at discrete or continuous time instants.
  - this defines discrete- or continuous-time random processes.
- Sample function values may take on discrete or continuous values.
  - this defines discrete- or continuous-parameter random processes.
Random Processes

Random processes can be described by functions $X_1(t)$, $X_2(t)$, and $X_ξ(t)$, where $t$ represents time. The sample space $S$ contains points $s_1$, $s_2$, and $s_ξ$. Each point in the sample space corresponds to a different random process.

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Random Processes vs. Random Variables

What is the difference between a random variable and a random processes?

- For a random variable, the outcome of a random experiment is mapped onto a *variable*, e.g., a number.
- For a random processes, the outcome of a random experiment is mapped onto a *waveform* that is a function of time.

Suppose that we observe a random process $X(t)$ at some time $t_1$ to generate the observation $X(t_1)$ and that the number of possible sample functions or waveforms, $\zeta$, is finite.

If $X_i(t_1)$ is observed with probability $P_i$, then the collection of numbers $\{X_i(t_1)\}$, $i = 1, 2, \ldots, \zeta$ forms a random variable, denoted by $X(t_1)$, having the probability distribution $P_i, i = 1, 2, \ldots, \zeta$. 
Random Processes

The collection of \(n\) random variables, \(X(t_1), \ldots, X(t_n)\), has the joint cdf
\[
F_{X(t_1), \ldots, X(t_n)}(x_1, \ldots, x_n) = P_r(X(t_1) < x_1, \ldots, X(t_n) < x_n).
\]
A more compact notation can be obtained by defining the vectors
\[
\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \\
\mathbf{X}(t) = (X(t_1), X(t_2), \ldots, X(t_n))^T
\]
Then the joint cdf and joint pdf of \(\mathbf{X}(t)\) are, respectively,
\[
F_{\mathbf{X}(t)}(\mathbf{x}) = P(\mathbf{X}(t) \leq \mathbf{x}) \\
f_{\mathbf{X}(t)}(\mathbf{x}) = \frac{\partial^n F_{\mathbf{X}(t)}(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_n}
\]
A random process is **strictly stationary** if and only if the equality
\[
f_{\mathbf{X}(t)}(\mathbf{x}) = f_{\mathbf{X}(t+\tau)}(\mathbf{x})
\]
holds for all sets of time instants \(\{t_1, t_2, \ldots, t_n\}\) and all time shifts \(\tau\).
Ensemble and Time Averages

For a random process, we define the following two operators

\[ E[\cdot] \triangleq \text{ensemble average} \]
\[ \langle \cdot \rangle \triangleq \text{time average} \]

The ensemble mean or ensemble average of a random process \( X(t) \) at time \( t \) is

\[ \mu_X(t) \equiv E[X(t)] = \int_{-\infty}^{\infty} x f_X(t)(x)dx \]

The time average mean or time average of a random process \( X(t) \) is

\[ \langle X(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt \]

- In general, the time average mean \( \langle X(t) \rangle \) is also a random variable, because it depends on the particular sample function that is observed or chosen for time averaging.
Example

Consider the random process shown below.

\[ X_1(t) = a \quad P_1 = 1/4 \]

\[ X_2(t) = 0 \quad P_2 = 1/2 \]

\[ X_3(t) = -a \quad P_3 = 1/4 \]
Example

The *ensemble mean* is

\[
E[X(t)] = X_1(t)P_1 + X_2(t)P_2 + X_3(t)P_3 \\
= a \cdot 1/4 + 0 \cdot 1/2 + (-a) \cdot 1/4 = 0
\]

The *time average mean* is

\[
\langle X(t) \rangle = \begin{cases} 
  a & \text{with probability } 1/4 \\
  0 & \text{with probability } 1/2 \\
  -a & \text{with probability } 1/4 
\end{cases}
\]

Note that \( \langle X(t) \rangle \) is a random variable (since it depends on the sample function that is chosen for time averaging, while \( E[X(t)] \) is just a number (that in the above example is not a function of time \( t \), but in general may a function of the time variable \( t \)).
Moments and Correlations

E[·] = ensemble average operator.

[Ensemble] Mean: \( \mu_X(t_1) = E[X(t_1)] = \int_{-\infty}^{\infty} x f_{X(t_1)}(x)dx \)

[Ensemble] Variance:
\[
\sigma^2_X(t_1) = E[(X(t_1) - \mu_X(t_1))^2] = \int_{-\infty}^{\infty} (x - \mu_X(t_1))^2 f_{X(t_1)}(x)dx
\]

[Ensemble] Autocorrelation: \( \phi_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] \)

[Ensemble] Autocovariance:
\[
\mu_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] = \phi_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)
\]

If \( X(t) \) has zero mean, then \( \mu_{XX}(t_1, t_2) = \phi_{XX}(t_1, t_2) \).
Consider the random process

\[ X(t) = A \cos(2\pi f_c t + \Theta) \]

where \( A \) and \( f_c \) are constants. The phase \( \Theta \) is assumed to be a uniformly distributed random variable with pdf

\[ f_{\Theta}(\theta) = \begin{cases} 
1/(2\pi), & 0 \leq \theta \leq 2\pi \\
0, & \text{elsewhere}
\end{cases} \]

The ensemble mean of \( X(t_1) \) is obtained by averaging over the pdf of \( \Theta \):

\[
\mu_X(t_1) = E_{\Theta}[X(t_1)] = E_{\Theta}[A \cos(2\pi f_c t_1 + \Theta)] \\
= \int_0^{2\pi} A \cos(2\pi f_c t_1 + \theta) f_{\Theta}(\theta) d\theta \\
= \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f_c t_1 + \theta) d\theta \\
= \frac{A}{2\pi} \sin(2\pi f_c t_1 + \theta) \bigg|_0^{2\pi} = 0
\]
Example (cont’d)

The autocorrelation of \( X(t) = A \cos(2 \pi f_c t + \Theta) \) is

\[ \phi_{XX}(t_1, t_2) = \mathbb{E}_\Theta[X(t_1)X(t_2)] \]
\[ = \mathbb{E}_\Theta[A^2 \cos(2 \pi f_c t_1 + \Theta) \cos(2 \pi f_c t_2 + \Theta)] \]
\[ = \frac{A^2}{2} \mathbb{E}_\Theta[\cos(2 \pi f_c t_1 + 2 \pi f_c t_2 + 2 \Theta)] + \frac{A^2}{2} \mathbb{E}_\Theta[\cos(2 \pi f_c (t_1 - t_2))] \]

But

\[ \mathbb{E}_\Theta[\cos(2 \pi f_c t_1 + 2 \pi f_c t_2 + 2 \Theta)] = \frac{1}{2 \pi} \int_0^{2 \pi} \cos(2 \pi f_c t_1 + 2 \pi f_c t_2 + 2 \theta) d\theta \]
\[ = \frac{1}{4 \pi} \sin(2 \pi f_c t_1 + 2 \pi f_c t_2 + 2 \theta) \bigg|_0^{2 \pi} \]
\[ = 0 \]

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Example (cont’d)

Also,
\[ E_\Theta[\cos(2\pi f_c(t_1 - t_2))] = \cos 2\pi f_c(t_1 - t_2) \]

Hence,
\[
\phi_{XX}(t_1, t_2) = A^2 \cos 2\pi f_c(t_1 - t_2)
\]
\[
= \frac{A^2}{2} \cos 2\pi f_c \tau, \quad \tau = t_1 - t_2
\]

The autocovariance of \( X(t) \) is
\[
\mu_{XX}(t_1, t_2) = \phi_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)
\]
\[
= \phi_{XX}(\tau)
\]

since \( \mu_X(t) = 0 \).
A **wide sense stationary** random process $X(t)$ has the property

$$\mu_X(t) = \mu_X \quad \text{a constant}$$

$$\phi_{XX}(t_1, t_2) = \phi_{XX}(\tau) \quad \text{where } \tau = t_1 - t_2$$

The autocorrelation function only depends on the time difference $\tau$.

If a random process is strictly stationary, then it is wide sense stationary. **The converse is not true.**

strictly stationary $\longrightarrow$ wide sense stationary

For a Gaussian random process only

strictly stationary $\longleftrightarrow$ wide sense stationary

The previous example is a wide sense stationary random process.
Some Properties of $\phi_{XX}(\tau)$

The autocorrelation function, $\phi_{XX}(\tau)$, of a wide sense stationary random process $X(t)$ satisfies the following properties.

1. $\phi_{XX}(0) = E[X^2(t)]:$ total power ac + dc
2. $\phi_{XX}(\tau) = \phi_{XX}(-\tau):$ autocorrelation is an even function
3. $|\phi_{XX}(\tau)| \leq \phi_{XX}(0):$ a variant of the Cauchy-Schwartz inequality. Proof on next slide.
4. $\phi_{XX}(\infty) = E^2[X(t)] = \mu_X^2:$ dc power, if $X(t)$ has no periodic components.
5. If $X(t)$ is periodic with period $T$, then $\phi_{XX}(\tau)$ is periodic with the same period $T$.
6. If $\mu_X(t) = \mu_X(t + T)$ and $\phi_{XX}(t, \tau) = \phi_{XX}(t + T, \tau)$ the random process is said to be periodic wide sense stationary or cyclostationary. Digitally modulated waveforms are cyclostationary random processes.
The inequality $|\phi_{XX}(\tau)| \leq \phi_{XX}(0)$ can be established through the following steps.

\[
0 \leq E[(X(t + \tau) \pm X(t))^2] = E[X^2(t) + X^2(t + \tau) \pm 2X(t + \tau)X(t)] = E[X^2(t)] + E[X^2(t + \tau)] \pm 2E[X(t + \tau)X(t)] = 2E[X^2(t)] \pm 2E[X(t + \tau)X(t)] = 2\phi_{XX}(0) \pm 2\phi_{XX}(\tau) .
\]

Therefore,

\[
\pm \phi_{XX}(\tau) \leq \phi_{XX}(0) \quad \text{and} \quad |\phi_{XX}(\tau)| \leq \phi_{XX}(0) .
\]