GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING
Final Examination - Fall 2013
EE 4601: Communication Systems

Aids Allowed: Course textbook
one $8\frac{1}{2} \times 11$ two-side crib sheet
Attempt all questions
All questions are of equal value

DATE: Friday December 13, 2013.

TIME: 2:50pm - 5:40pm
INSTRUCTOR: Professor G.L. Stüber

Solutions

1) **Random Processes:** Let $X(t)$ be a Gaussian random process with zero mean and autocorrelation function

$$\phi_{XX}(\tau) = 10^{-5} B_w \text{sinc}^2(B_w \tau), \qquad B_w = 10^6 \text{ Hz}$$

The process $X(t)$ is input to a linear filter having the impulse response

$$h(t) = B\text{sinc}(2Bt), \qquad B = 10^5 \text{ Hz}$$

and the output process is $Y(t) = X(t) * h(t)$.

a) **4 marks:** Sketch and fully dimension the power spectral density of $Y(t)$.

b) **2 marks:** What is the probability density function of the random variable $Y(t_1)$ at any time instant t_1?

c) **4 marks:** What is the probability that the output $Y(t_1)$ lies in the range $[-0.1, 0.1]$?

a) From tables

$$\begin{cases} 1 - \frac{|t|}{T}, & |t| \le T \\ 0, & |t| > T \end{cases} \Longleftrightarrow T\,\text{sinc}^2(fT)$$

and duality

$$g(t) \Longleftrightarrow G(f)$$
$$G(t) \Longleftrightarrow g(-f)$$

we have

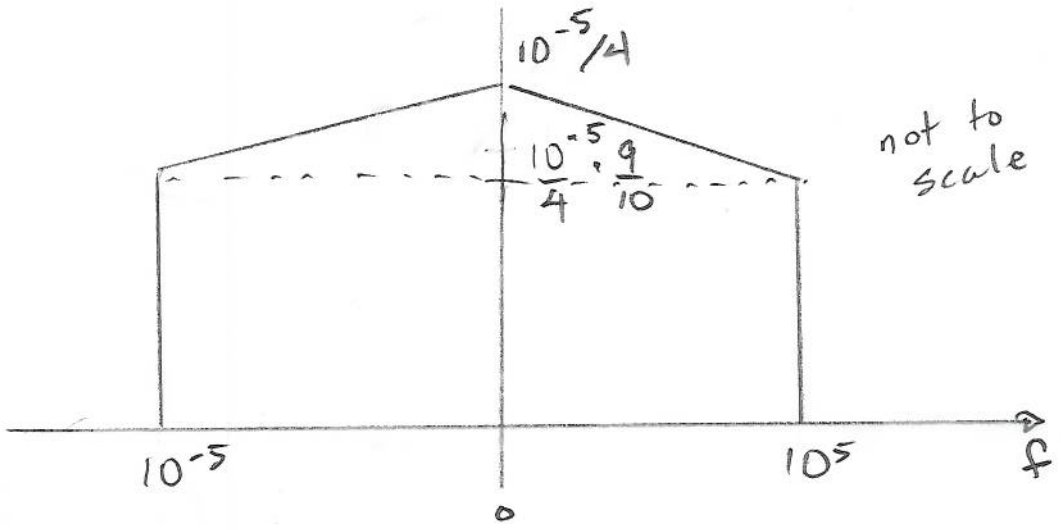
$$\Phi_{ss}(f) = \mathcal{F}\{\phi_{ss}(\tau)\} = \begin{cases} 10^{-5}\left(1 - \frac{|f|}{10^6}\right), & |f| \le 10^6 \\ 0, & |f| > 10^6 \end{cases}$$

Also,

$$H(f) = \frac{1}{2}\,\text{rect}\left(\frac{f}{2B}\right)$$

Then

$$\Phi_{yy}(f) = |H(f)|^2 \Phi_{xx}(f) = \begin{cases} \frac{10^{-5}}{4}\left(1 - \frac{|f|}{10^6}\right), & |f| \le 10^5 \\ 0, & |f| > 10^5 \end{cases}$$



b) $y(t_1) \sim N(0, \sigma^2)$

$$\sigma^2 = \int_{-\infty}^{\infty} \Phi_{YY}(f)\,df$$

$$= \frac{10^{-5}}{4} \times 10^6 - \frac{10^{-5}}{4} \cdot \frac{9}{10}\left(10^6 - 10^5\right)$$

$$= \frac{10^{-5} \times 10^6}{4}\left(1 - \frac{9}{10} + \frac{9}{100}\right)$$

$$= \frac{10}{4} \cdot \frac{19}{100} = \frac{19}{40}$$

c) $P\left[-0.1 \le y(t_1) \le 0.1\right]$

$$= 1 - 2Q\left(\frac{0.1}{\sigma}\right)$$

$$= 1 - 2Q(.15)$$

$$= 0.12$$

2) ISI Channels: Consider the transmission of a binary data sequence $\{a_n\}$, where the a_n take on values from the set $\{-1+1\}$ with equal probability, using binary antipodal signaling. The overall pulse response is $p(t) = g(t) * c(t) * h(t)$, where $g(t)$ is the transmit filter, $c(t)$ is the channel, and $h(t)$ is the receiver matched filter. The sampled pulse $p_k = p(kT)$ is such that

$$p_k = \begin{cases} 1, & k = 0 \\ 0.25, & k = 1 \\ 0.25, & k = 2 \\ 0, & \text{elsewhere} \end{cases}$$

The samples at the output of the receiver matched filter are

$$y_n = p_0 a_n + \sum_{k \neq n} a_k p_{n-k} + n_n$$

where the n_n are independent zero-mean Gaussian random variables with variance σ^2. A decision on a_n is made by only using the observation y_n, i.e., no equalization is used.

a) **4 marks:** What are the possible values for the ISI term and their probabilities?

b) **4 marks:** Derive an expression for the probability of bit error.

c) **2 marks:** Now suppose that a zero forcing equalizer is used. Would the zero-forcing solution minimize the peak distortion after equalization?

a) $y_n = a_n + \underbrace{0.25(a_{n-1} + a_{n-2})}_{ISI} + n_n$

ISI	Prob
-0.5	$1/4$
0	$1/4$
0.5	$1/4$

b) Suppose $a_n = 1$

$$y_n = 1 + 0.25(a_{n-1} + a_{n-2}) + n_n$$

$$P_b = \frac{1}{4} Q\left(\frac{0.5}{\sigma}\right) + \frac{1}{2} Q\left(\frac{1}{\sigma}\right) + \frac{1}{4} Q\left(\frac{1.5}{\sigma}\right)$$

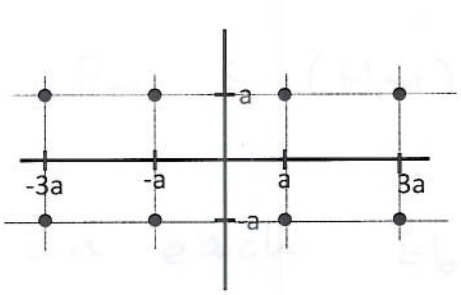
c)

$$D_p = \frac{1}{|P_0|} \sum_{n \neq 0} |P_n|$$

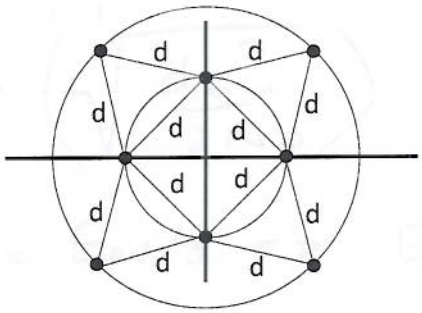
$$= 0.5$$

Since $D_p < 1$, the zero forcing solution minimizes the peak distortion after equalization.

3) **Signal Constellations:** Consider the two different signal constellations shown below, where each constellation has 8 points.



(A)



(B)

a) **5 marks:** Find the minimum distance $d_{\min}$ of each signal constellation in terms of the average symbol energy E_{av}.

b) **3 marks:** Using $d_{\min}$, derive a simple upper bound on the probability of *symbol* error for each constellation in terms of the *bit* energy-to-noise ration E_b/N_o.

c) **2 marks:** At high E_b/N_o, which constellation will be better in terms of error probability performance? How much better will it be in terms of the E_b/N_o (in dB) required to give the same error probability as the other constellation?

a) (A) $E_{av} = \frac{4}{8}(a^2+a^2) + \frac{4}{8}(a^2+9a^2)$

$= \frac{4}{8}(12a^2) = 6a^2 \Rightarrow a = \sqrt{E_{av}/6}$

$d_{\min_A} = 2a = 2\sqrt{E_{av}/6} = \sqrt{2E_{av}/3}$

(B) $E_{av} = \frac{4}{8}\left(\frac{d^2}{2}\right) + \frac{4}{8}\left(\frac{d}{2} + \frac{\sqrt{3}}{2}d\right)^2$

$= \frac{4}{8}\left(\frac{d^2}{2} + \frac{d^2}{4} + 2\frac{\sqrt{3}}{4}d^2 + \frac{3}{4}d^2\right)$

$= \frac{4}{8}\left(\frac{6d^2}{4} + 2\frac{\sqrt{3}}{4}d^2\right) = \frac{3+\sqrt{3}}{4}d^2$

$d_{\min_B} = \sqrt{\frac{4E_{av}}{3+\sqrt{3}}}$

b) We have

$$P_M \leq (M-1)Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$$

In each $E_b = E_{av}/3 \Rightarrow E_{av} = 3E_b$

(A) $$P_M \leq 7Q\left(\sqrt{\frac{2E_{av}}{3.2N_0}}\right) = 7Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

(B) $$P_M \leq 7Q\left(\sqrt{\frac{4E_{av}}{(3+\sqrt{3})}\cdot\frac{1}{2N_0}}\right)$$

$$= 7Q\left(\sqrt{\frac{6}{3+\sqrt{3}}\frac{E_b}{N_0}}\right)$$

c) At high E_b/N_0, (B) is better by

$$10\log_{10}\left(\frac{6}{3+\sqrt{3}}\right) = 1.03 \text{ dB}$$

4) **Modulation:** Two data streams, $\{x_n^1\}$ and $\{x_n^2\}$, $x_n^1, x_n^2 \in \{-1, +1\}$, are to be transmitted using *unbalanced* QPSK, where the data rate for $\{x_n^1\}$ is 10 kbps and that for $\{x_n^2\}$ is 1 Mbps. The unbalanced QPSK bandpass waveform is given by

$$s(t) = A_1 \sum_n x_n^1 u_{T_1}(t - nT_1)\cos(2\pi f_c t) + A_2 \sum_n x_n^2 u_{T_2}(t - nT_2)\sin(2\pi f_c t)$$

where $u_{T_i}(t) = u(t) - u(t - T_i)$, $u(t)$ is the unit step function, and n is the baud symbol index.

a) **5 marks:** Relate the amplitudes of the waveforms, A_1 and A_2, such that both data streams have equal energies per bit.

b) **5 marks:** With A_1 and A_2 so related, find the possible phases that the bandpass waveform can assume.

a) For stream $\{x_n^1\}$

$$E_1 = \frac{A_1^2 T_1}{2}$$

For stream $\{x_n^2\}$

$$E_2 = \frac{A_2^2 T_2}{2}$$

For equal energy per bit

$$A_1^2 T_1 = A_2^2 T_2$$

$$\frac{A_2}{A_1} = \sqrt{\frac{T_1}{T_2}} = \sqrt{\frac{10^6}{10^4}} = 10$$

b) At any time t, the bandpass waveform has the form

$$s(t) = \pm A_1 \cos 2\pi f_c t \pm A_2 \sin 2\pi f_c t$$

$$= \pm A \cos(2\pi f_c t \pm \phi)$$

$$= A \cos(2\pi f_c t \pm \phi \pm \pi)$$

$$\phi = \tan^{-1} \frac{A_2}{A_1} = \tan^{-1}(10) = .46827\pi$$

$$\therefore \phi \in \{ \pm .46827\pi, \pm .53173\pi \}$$

5) **Error Probability:** Suppose that the following signal vectors are used on an AWGN channel with two-sided noise power spectral density of $N_o/2$ watts/Hz.

$$\begin{aligned} \mathbf{s}_1 &= (a, 2a, a) \\ \mathbf{s}_2 &= (a, 0, a) \\ \mathbf{s}_3 &= (-a, 2a, a) \\ \mathbf{s}_4 &= (-a, 0, a) \end{aligned}$$

a) (**5 marks**) Find the *exact* probability of *symbol* error in terms of the average modulated *symbol* energy-to-noise ratio. *Hint: Use a graphical approach to visualize the signal vectors.*

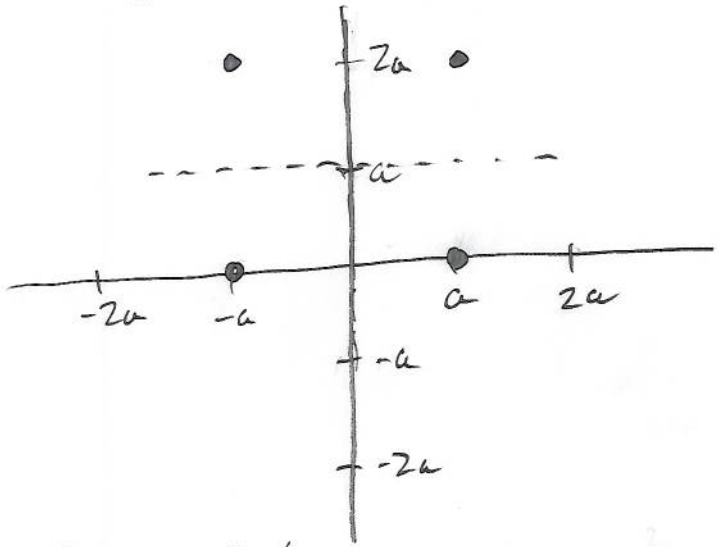
b) (**3 marks**) Suppose that a new set of signal vectors is constructed by translating the vectors as follows:

$$\hat{\mathbf{s}}_i = \mathbf{s}_i - \mathbf{b}\ , \quad i = 1, \ldots 4$$

What vector $\mathbf{b}$ will minimize the average *symbol* energy in the new signal constellation $\{\hat{\mathbf{s}}_i\}_{i=1}^4$?

c) (**2 marks**) How would the symbol error probability in part b) compare with that in part a)?

a) Ignoring the third co-ordinate the signal vectors are plotted below



$\sigma^2 = \frac{N_0}{2}$

let $Q = Q\left(\frac{a}{\sigma}\right)$

$P_{correct} = (1 - Q)^2$

$P_M = 2Q - Q^2$

$$E_{av} = \frac{1}{4}\left(a^2 + 4a^2 + a^2 + a^2 + a^2 + a^2 + 4a^2 + a^2 + a^2 + a^2 \right) = 14a^2$$

$$a = \sqrt{E_{av}/14}$$

$$Q = Q\left(\sqrt{\frac{E_{av}}{14} \cdot \frac{2}{N_0}}\right) = Q\left(\sqrt{\frac{1}{7}\frac{E_{av}}{N_0}}\right)$$

$$P_M = 2Q - Q^2$$

b) Move the center of the signal constellation to the origin

$$b = (0, -a, -a)$$

$$\hat{E}_{av} = 2a^2 \qquad a = \sqrt{E_{av}/2}$$

$$P_M = 2Q - Q^2$$

$$Q = Q\left(\sqrt{\frac{E_{av}}{2} \cdot \frac{2}{N_0}}\right) = Q\left(\sqrt{\frac{E_{av}}{N_0}}\right)$$

c) Signal constellation in part b) requires $10\log_{10}7 = 8.45$ dB less E_{av}/N_0 for the same P_M in part a).