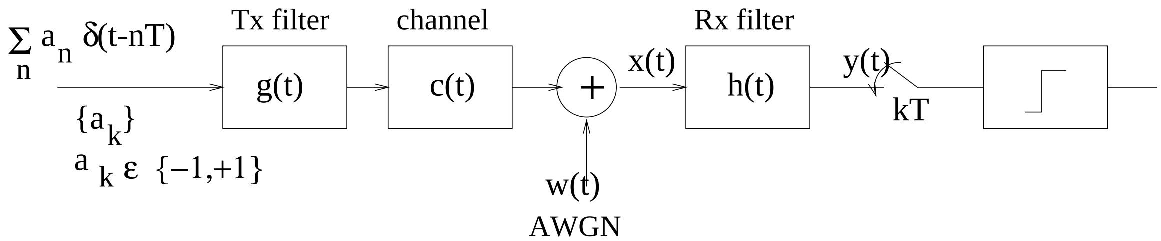
EE4061
Communication Systems

Week 10

Intersymbol Interference

Nyquist Pulse Shaping

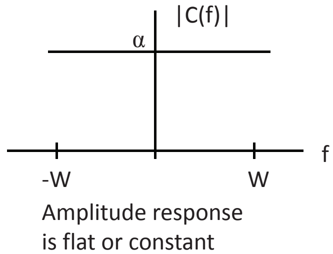
Intersymbol Interference (ISI)

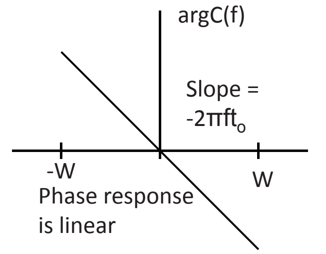


An ideal channel $c(t)$ only scales and time shifts the signal $g(t)$, but otherwise leaves it undistorted, i.e. for an ideal channel

$$
\begin{aligned}
c(t) &= \alpha\delta(t - t_o) \\
g(t) * c(t) &= \alpha g(t - t_o) \\
C(f) &= \alpha e^{-j2\pi f t_o}
\end{aligned}
$$

ISI





Let

$$p(t) = g(t) * c(t) * h(t)$$
$$\updownarrow$$
$$P(f) = G(f)C(f)H(f)$$

Then $y(t) = \sum_n a_n p(t - nT) + n(t)$, where $n(t) = w(t) * h(t)$

ISI

$$\begin{aligned} y_i = y(iT) &= \sum_n a_n p((i-n)T) + n(iT) \\ &= \sum_n a_n p_{i-n} + n_i \end{aligned}$$

where $p_{i-n} = p((i-n)T)$, $n_i = nT$

$$y_i = a_i p_0 + \sum_{n \neq i} a_n p_{i-n} + n_i$$

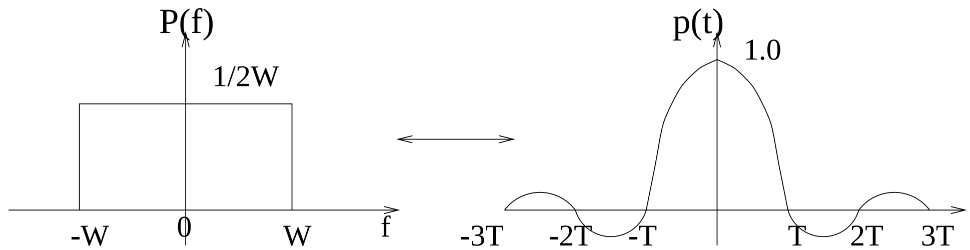
$a_i p_o-$ desired term,
$\sum_{n \neq i} a_n p_{i-n}-$ISI
n_i- noise
In the absence of ISI and noise, $y_i = a_i p_0$. Any pulse $p(t)$ with

$$p_i = p(iT) = \left\{ \begin{array}{ll} p_0 & i = 0 \\ 0 & i \neq 0 \end{array} \right. = p_0 \delta_{i0}$$

yields zero ISI

Bandlimited Pulse Shaping

What pulse shapes $p(t)$, $p(t) = g(t) * c(t) * h(t)$ yield zero ISI.
Suppose $P(f) = \frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$ where $W = 1/2T = R/2$, T is the baud duration, R is the baud rate

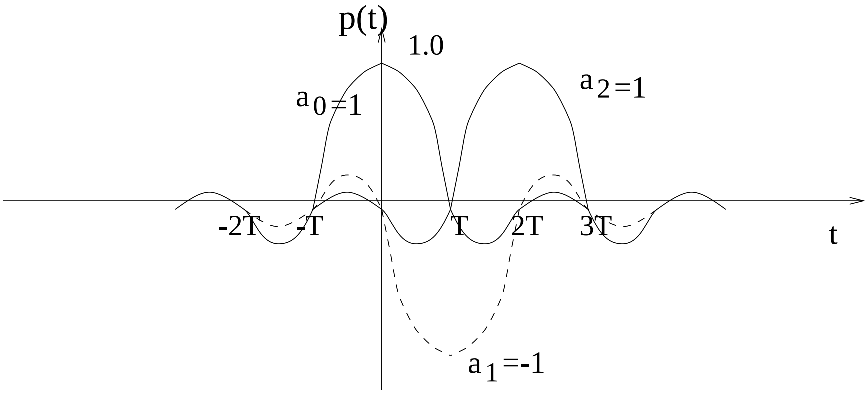


$p(t) = \mathrm{sinc}(2Wt) = \mathrm{sinc}(t/T)$, $T = 1/2W$
Note that

$$\begin{aligned} p_i = p(iT) &= \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases} \\ &= \delta_{i0} \end{aligned}$$

This pulse results in zero ISI. Note that $p(t)$ is noncausal

ISI - Problems



Problems:

1. $P(f) = \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$ is not realizable
2. $p(t)$ decays slowly with time. It decreases with $1/|t|$ for large t. Therefore, it is very sensitive to sampler phase

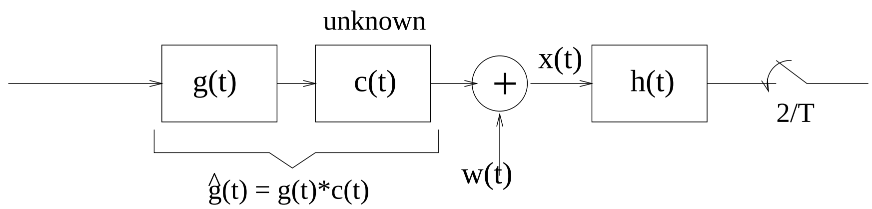
Problems with 'sinc' pulse

$$
\begin{aligned}
y(t) &= \sum_n a_n p(t - nT) \\
y(\Delta t) &= \sum_n a_n p(\Delta t - nT) \\
&= \sum_n a_n \frac{\sin[\pi(\Delta t - nT)/T]}{\pi(\Delta t - nT)/T} \\
&= \sum_n \frac{\sin(\pi \Delta t/T)\cos(\pi n) - \cos(\pi \Delta t/T)\sin(\pi n)}{\pi \Delta t/T - n\pi} \\
&= \sum_n a_n \frac{(-1)^n \sin(\pi \Delta t/T)}{\pi \Delta t/T - n\pi} \\
&= a_0 \text{sinc}(\Delta t/T) + \frac{\sin(\pi \Delta t/T)}{\pi} \sum_{n \neq 0} \frac{a_n (-1)^n}{\Delta t/T - n}
\end{aligned}
$$

Last term is not absolutely summable.

We have seen $y_i = y(iT) = a_i p_0 + \sum_{n \neq i} a_n p_{i-n} + n_i$
where $p_k = p(kT)$, $n_i = n(iT)$.

Matched Filtering and Pulse Shaping



- To maximize the signal-to-noise ratio at the output of the receiver filter $h(t)$, in theory we match the receiver filter to the received pulse $\hat{g}(t) = g(t) * c(t)$, i.e., $h(t) = \tilde{g}(T - t)$. However, if $c(t)$ is unknown, then so is $h(t)$.
- <u>Practical Solution:</u> Choose $h(t)$ matched to the transmitted pulse $g(t)$, i.e., choose $h(t) = g(T - t)$, over-sample by a factor of 2, and process 2 samples per baud interval.
  - This is optimal, similar to the case when $c(t)$ is known, but the proof is beyond the scope of this course.

Matched Filtering and Pulse Shaping

- To design the transmit and receiver filters, we will assume an ideal channel $c(t) = \delta(t)$, so that the overall pulse (ignoring time delay) is

$$\begin{aligned} p(t) &= g(t) * h(t) \\ &= g(t) * g(-t) \end{aligned}$$

- Taking the Fourier transform of both sides

$$P(f) = G(f)G^*(f) = |G(f)|^2$$

- Hence

$$|G(f)| = \sqrt{|P(f)|}$$

- For many practical pulses, $g(t)$, we will also see that $g(t) = g(-t)$, i.e., the pulse is even in t, so that $h(t) = g(t)$.

Conditions for ISI free transmission

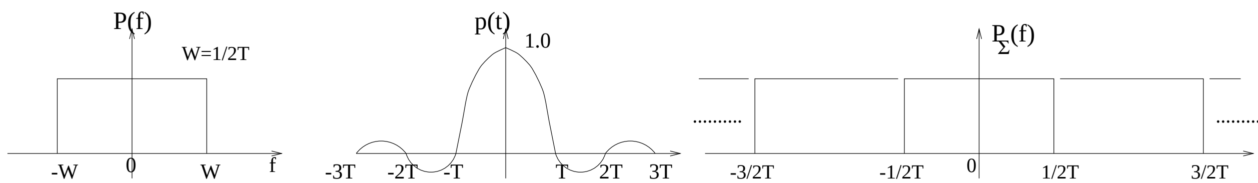
The condition for ISI-free transmission is

$$p_k = \delta_{k0} p_0 = \begin{cases} p_0 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

That is, $p(t)$ must have equally spaced zero crossings, separated by T seconds.
Theorem: The pulse $p(t)$ satisfies $p_k = \delta_{k0} p_0$ iff

$$P_{\sum}(f) \triangleq \frac{1}{T} \sum_{n=-\infty}^{\infty} P(f + n/T) = p_0$$

That is the folded spectrum $P_{\sum}(f)$ is flat.



ISI free transmission

Proof:

$$\begin{aligned} p_k &= \int_{-\infty}^{\infty} P(f) e^{j2\pi f k T} df \\ &= \sum_{n=-\infty}^{\infty} \int_{(2n-1)/2T}^{(2n+1)/2T} P(f) e^{j2\pi f k T} df \quad f' = f - n/T \\ &= \sum_{n=-\infty}^{\infty} \int_{-1/2T}^{1/2T} P(f' + n/T) e^{j2\pi k (f' + n/T) T} df' \\ &= \int_{-1/2T}^{1/2T} e^{j2\pi f' k T} \left[\sum_{n=-\infty}^{n=\infty} P(f' + n/T) \right] df' \end{aligned} \tag{1}$$

To prove sufficiency, we assume that $\sum_{n=-\infty}^{\infty} P(f' + n/T) = p_0 T$ is true. Then,

$$p_k = p_0 T \int_{-1/2T}^{1/2T} e^{j2\pi f' k T} df' = \frac{\sin \pi k}{\pi k} p_0 = \delta_{k0} p_{k0}$$

To prove necessity, we have from (1)

$$p_k = T \int_{-1/2T}^{1/2T} P_{\Sigma}(f') e^{j2\pi f' k T} df'$$

Nyquist Pulse

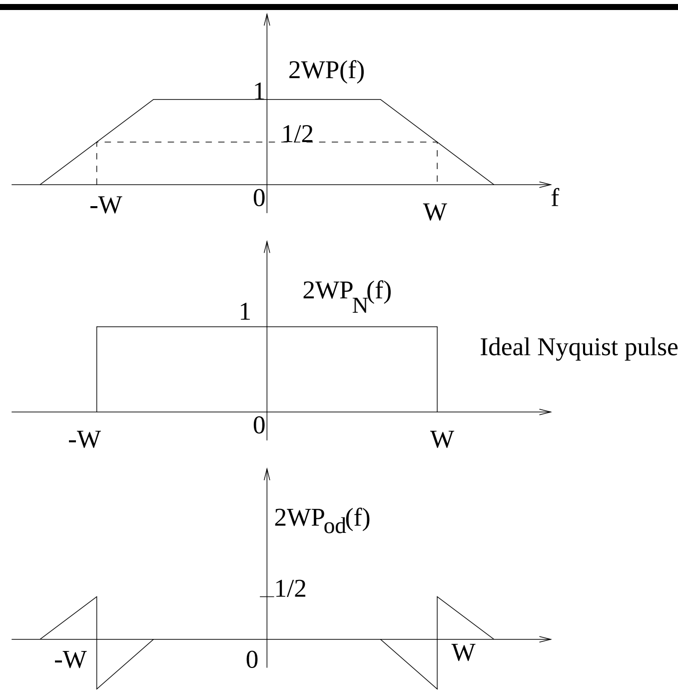
Hence, p_k and $P_\Sigma(f)$ are a Fourier series pair, i.e.,

$$P_\Sigma(f) = \sum_{k=-\infty}^{\infty} p_k e^{j2\pi f kT}$$

If $p_k = p_0 \delta_{k0}$ is assumed true, then from the above equation $P_\Sigma(f) = p_0$.

- Nyquist Pulse Shaping: A pulse $p(t)$ that yields zero-ISI is one having a folded spectrum that is flat.
  - The pulse $p(t)$ can be generated by choosing $P(f)$ as shown on the following slide.

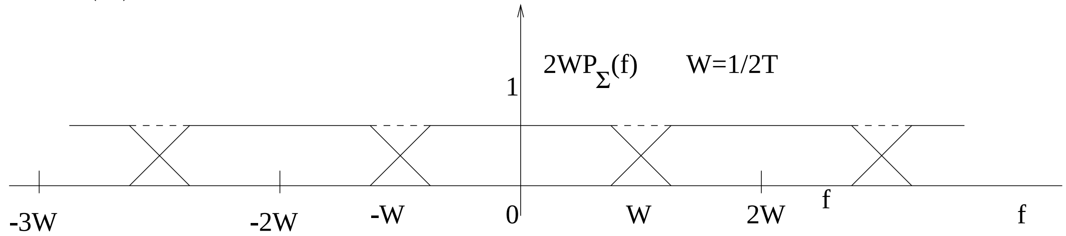
Nyquist Pulse Shaping



Note $P(f) = P_N(f) + P_{\rm od}(f)$.
$P_{\rm od}(f)$ can be any function that has skew symmetry about $f = W$.

Nyquist Pulse

Note that $P_\Sigma(f)$ is flat under this condition.



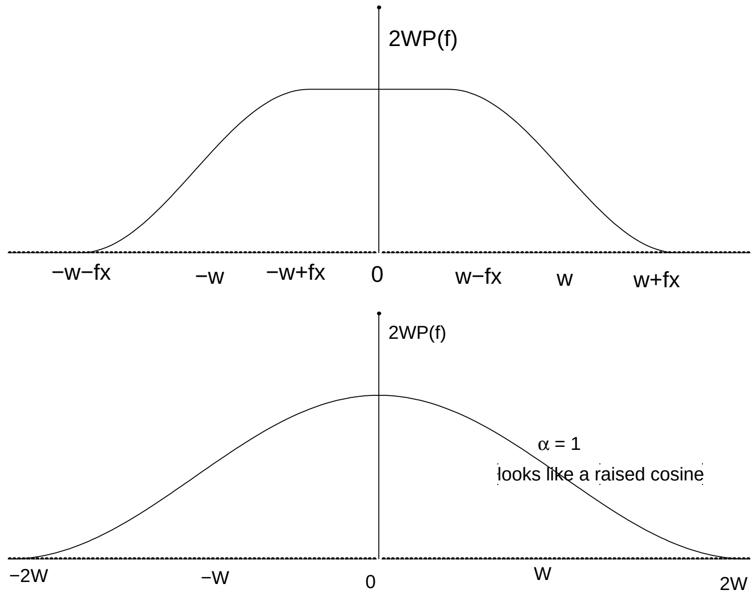
Example: Raised Cosine

$$2WP_{\rm od}(f) = \begin{cases} -\frac{1}{2} - \frac{1}{2}\sin\frac{\pi(|f|-W)}{2f_x} & W - f_x \le |f| \le W \\ \frac{1}{2} - \frac{1}{2}\sin\frac{\pi(|f|-W)}{2f_x} & W \le |f| \le W + f_x \end{cases}$$

f_x = bandwidth expansion, $\frac{f_x}{W} \times 100$ = excess bandwidth (%), $\alpha = \frac{f_x}{W}$ = roll off factor

$$2WP(f) = \begin{cases} 1 & 0 \le |f| \le W - f_x \\ \frac{1}{2}\left[1 - \sin\frac{\pi(|f|-W)}{2f_x}\right] & W - f_x \le |f| \le W + f_x \\ 0 & |f| \ge W + f_x \end{cases}$$

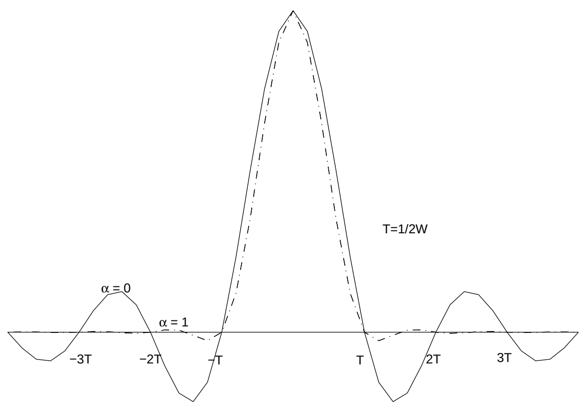
Raised Cosine Pulse



Raised Cosine Impulse Response

Impulse response - Since $P(f)$ is even, the inverse cosine transform yields

$$
\begin{aligned}
p(t) &= 2\int_0^{W+f_x} P(f)\cos 2\pi f t df \\
&= 2.\frac{1}{2W}\int_0^{W-f_x}\cos 2\pi f t df + 2.\frac{1}{2W}\int_{W-f_x}^{W+f_x}\frac{1}{2}\left[1-\sin\frac{\pi|f|-W}{2f_x}\right]\cos 2\pi f t df \\
&= \frac{\sin 2\pi W t}{2\pi W t}.\frac{\cos 2\pi f_x t}{1-(4f_x t)^2}
\end{aligned}
$$



Square Root Raised Cosine Pulse

- To implement a matched filter, we split the overall pulse $P(f)$ between the transmit and receive filters, i.e., $p(t) = g(t) * g(-t)$.
- We have seen earlier that $P(f) = G(f)G^*(f) = |G(f)|^2$, so that $|G(f)| = \sqrt{P(f)}$.
- With square-root raised cosine pulse shaping

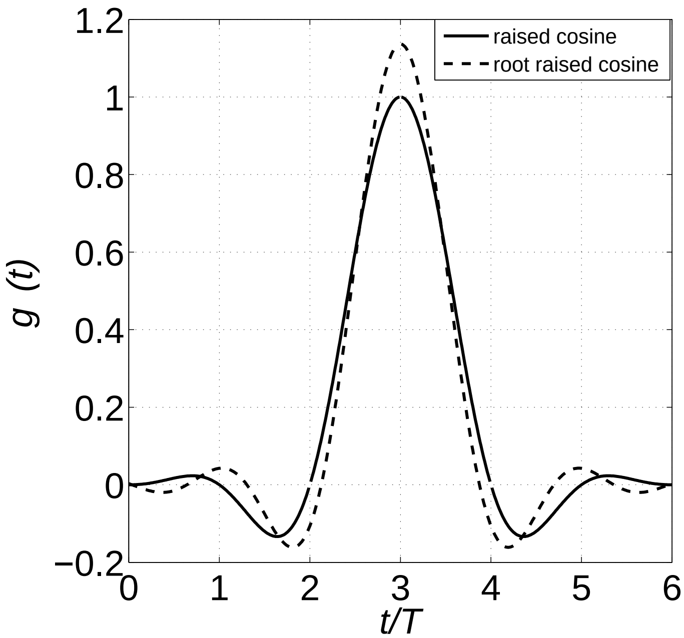
$$\sqrt{2W}|G(f)| = \begin{cases} 1 & 0 \le |f| \le wW - f_x \\ \sqrt{\frac{1}{2}\left[1 - \sin\frac{\pi(|f|-W)}{2f_x}\right]} & W - f_x \le |f| \le W + f_x \\ 0 & |f| \ge W + f_x \end{cases}$$

- The impulse response is

$$g(t) = 4\alpha \frac{\cos\left[(1+\alpha)\pi t/T\right] + \sin\left[(1-\alpha)\pi t/T\right](4\alpha t/T)^{-1}}{\pi\sqrt{T}\left[1 - 16\alpha^2 t^2/T^2\right]}$$

where $\alpha = f_x/W$.

Square Root Raised Cosine Pulse



Raised cosine and root raised cosine pulses with roll-off factor $\alpha = 0.5$. The pulses are truncated to length $6T$ and time shifted by $3T$ to yield causal pulses.

M-ary QAM

- Quadrature Amplitude Modulation (QAM), the transmitted waveform in each baud interval takes on one of the following M waveforms

$$s_m(t) = \sqrt{\frac{2E_0}{T}} g(t) \left(a_m^c \cos(2\pi f_c t) - a_m^s \sin(2\pi f_c t) \right)$$

  where

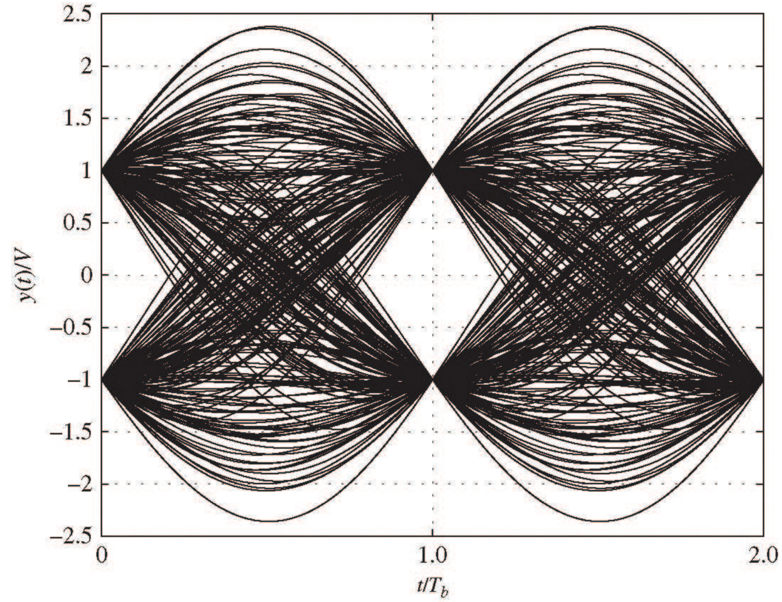
$$a_m^{\{c,s\}} \in \{\pm 1, \pm 3, \pm 5, \pm(M-1)\}$$

  and $2E_0$ is the energy of the signal with the lowest amplitude, i.e., when $a_m^c, a_m^s = \pm 1$.

- You have seen this before for the case $g(t) = u_T(t)$; however, practical systems will use the root-raised cosine pulse for $g(t)$. Note that we use the normalization,

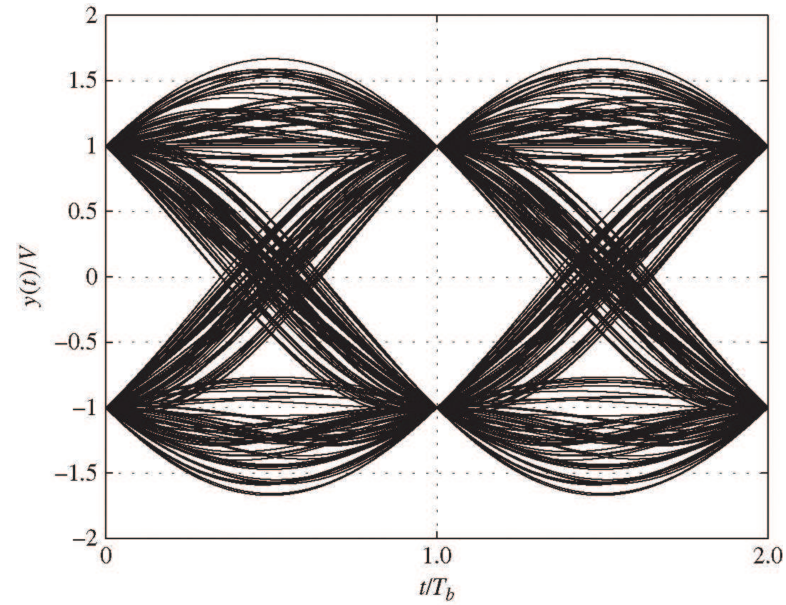
$$E_g = \int_{-\infty}^{\infty} g^2(t) dt = 1 .$$

Eye Diagram with Ideal Nyquist Pulse



Eye diagram when $P(f)$ is an ideal low pass filter.

Eye Diagram with Raised Cosine Pulse



Eye diagram when $P(f)$ is a raised cosine filter.