EE4601
Communication Systems

Week 9

QAM Error Probability

General Error Probability Analysis

QAM Signals

With QAM signals

$$s_m(t) = \sqrt{\frac{2E_0}{T}} a_m^c \cos(2\pi f_c t) - a_m^s \sin(2\pi f_c t) \qquad a_m^c, a_m^s \in \{\pm 1, \pm 3\}$$

The appropriate basis functions for the signal space are

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad f_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

Then

$$\begin{aligned} s_m(t) &= \sqrt{E_0} a_m^c f_1(t) + \sqrt{E_0} a_m^s f_2(t) \\ \mathbf{s}_m &= \sqrt{E_0}(a_m^c, a_m^s) \end{aligned}$$

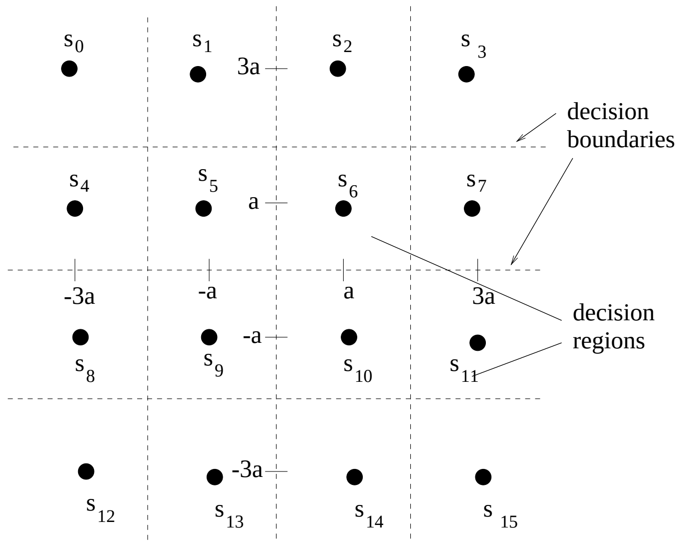
We randomly choose one of the 16 signals to transmit over an AWGN channel and receive $\mathbf{r} = \mathbf{s_m} + \mathbf{n}$, where $\mathbf{n} = (n_1, n_2)$, and the n_i are i.i.d. Gaussian random variables with variance $\sigma^2 = N_o/2$.

Our task is to find the probability of symbol error with minimum distance (or maximum likelihood) decisions.

QAM Signals

To calculate the probability of symbol error, we first must define appropriate *decision regions* by placing *decision boundaries* between the signal points. For 16-QAM this is shown below.

Note that $a = \sqrt{E_0}$ in the figure.



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QAM Signals

For problems of this type, and especially for one or two-dimensional signal spaces (this problem is 2-D), it is often easier to calculate the probability of correct reception.

For this problem there are 3 cases to consider, since we can observe graphically that

$$\begin{aligned}
P_{C|\mathbf{s}_5} &= P_{C|\mathbf{s}_6} = P_{C|\mathbf{s}_9} = P_{C|\mathbf{s}_{10}} \\
P_{C|\mathbf{s}_0} &= P_{C|\mathbf{s}_3} = P_{C|\mathbf{s}_{12}} = P_{C|\mathbf{s}_{15}} \\
P_{C|\mathbf{s}_1} &= P_{C|\mathbf{s}_2} = P_{C|\mathbf{s}_4} = P_{C|\mathbf{s}_7} = P_{C|\mathbf{s}_8} = P_{C|\mathbf{s}_{11}} = P_{C|\mathbf{s}_{13}} = P_{C|\mathbf{s}_{14}}
\end{aligned}$$

All these quantities can be expressed in terms of the parameter

$$Q \equiv Q\left(\frac{\sqrt{E_0}}{\sigma}\right) \qquad \sigma^2 = \frac{N_o}{2}$$

QAM Signals

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We have

$$\begin{aligned} P_{C|\mathbf{s}_5} &= (1-2Q)^2 = 1 - 4Q + 4Q^2 \\ P_{C|\mathbf{s}_0} &= (1-Q)^2 = 1 - 2Q + Q^2 \\ P_{C|\mathbf{s}_1} &= (1-Q)(1-2Q) = 1 - 3Q + 2Q^2 \end{aligned}$$

Then

$$\begin{aligned} P_C &= \frac{1}{4}P_{C|\mathbf{s}_5} + \frac{1}{4}P_{C|\mathbf{s}_0} + \frac{1}{2}P_{C|\mathbf{s}_1} \\ &= 1 - 3Q + \frac{9}{4}Q^2 \end{aligned}$$

Finally, the probability of error is $P_e = 1 - P_C = 3Q - \frac{9}{4}Q^2$

QAM Signals

Next, we need to find the *average* symbol energy. Remember that the energy in a symbol is equal to squared length of the signal vector.

In this case,

$$E_{\rm av} = \frac{1}{4}(E_0 + E_0) + \frac{1}{4}(9E_0 + 9E_0) + \frac{1}{2}(E_0 + 9E_0) = 10E_0$$

Hence, $E_0 = E_{\rm av}/10$, and

$$Q = Q\left(\frac{\sqrt{E_0}}{\sigma}\right) = Q\left(\sqrt{\frac{2E_0}{N_o}}\right) = Q\left(\sqrt{\frac{E_{\rm av}}{5N_o}}\right)$$

Finally,

$$P_e = 3Q\left(\sqrt{\frac{E_{\rm av}}{5N_o}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{E_{\rm av}}{5N_o}}\right)$$

where

$$\frac{E_{\rm av}}{N_o} = \text{average symbol energy-to-noise ratio}$$

QAM Signals

What about the bit error probability? That depends on the mapping of bits to symbols.

With Gray coding, a symbol error will usually result in one bit error. Certainly at most 4 bits errors will occur. Hence,

$$\frac{P_e}{4} \lessapprox P_b < P_e$$

Also, there are 4 bits per modulated symbol so that the average bit energy-to-noise ratio is

$$E_{b\ \mathrm{av}} = E_{\mathrm{av}}/4$$

So we can write

$$P_b \gtrapprox \frac{3}{4} Q\left(\sqrt{\frac{4}{5}\frac{E_{b\ \mathrm{av}}}{N_o}}\right) - \frac{9}{16} Q^2\left(\sqrt{\frac{4}{5}\frac{E_{b\ \mathrm{av}}}{N_o}}\right)$$

Binary Error Probability

Consider two signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$.

The received signal vector is

$$\mathbf{r} = \mathbf{s}_i + \mathbf{n}$$

A coherent maximum likelihood or minimum distance receiver decides in favor of the signal point $\mathbf{s}_1$ or $\mathbf{s}_2$ that is closest in Euclidean distance to the received signal point $\mathbf{r}$.

The error probability between $\mathbf{s}_1$ and $\mathbf{s}_2$ is

$$P(\mathbf{s}_1, \mathbf{s}_2) = Q\left(\sqrt{\frac{d_{12}^2}{2N_o}}\right)$$

where $d_{12}^2 = \|\mathbf{s}_1 - \mathbf{s}_2\|^2$ is the squared Euclidean distance between $\mathbf{s}_1$ and $\mathbf{s}_2$.

Error Probability and Euclidean Distance

The error probability depends on the *Euclidean distance* between the signal vectors.

If we have two signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, separated by Euclidean distance $d_{12} = \|\mathbf{s}_1 - \mathbf{s_2}\|$, then the error probability is

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_o}}\right)$$

For BPSK $d_{12} = 2\sqrt{E}$
For BFSK $d_{12} = \sqrt{2E}$

Voronoi Regions

Now suppose that we have a collection of M signal vectors, $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M$. The maximum likelihood receiver observes the received vector $\mathbf{r}$ and decides in favour of the signal vector that is closest in Euclidean distance (or squared Euclidean distance) to $\mathbf{r}$. That is

$$\hat{\mathbf{s}} = \mathrm{argmin}_{\mathbf{s}_i} \|\mathbf{r} - \mathbf{s}_i\|^2$$

The received signal vector lies in the N-dimensional Euclidean space $\mathbf{R}^N$. Suppose that we form M partitions of $\mathbf{R}^N$ in the following fashion

$$R_i = \{\mathbf{r} : \|\mathbf{r} - \mathbf{s}_i\| = \min_j \|\mathbf{r} - \mathbf{s}_j\|\}$$

The $R_i, i = 1, \ldots, M$ are called *Voronoi* regions.

The maximum likelihood decision can be put in the form

$$\hat{\mathbf{s}} = \mathbf{s}_i \text{ whenever } \mathbf{r} \in R_i$$

Error Probability

Under the assumption of equally likely transmitted symbols, the symbol error probability can be written as

$$P_M = 1 - P_C = 1 - \frac{1}{M}\sum_{j=1}^{M} P_{C|\mathbf{s_j}}$$

where $P_{C|\mathbf{s_j}}$ is the probability of a correct decision when $\mathbf{s}_j$ is sent.

The computation of P_M requires the set of probabilities $\{P_{C|\mathbf{s_j}}\}_{j=1}^{M}$.

However, a correct decision on $\mathbf{s}_j$ occurs if and only if the noise vector $\mathbf{n}$ does not move the received vector $\mathbf{r} = \mathbf{s}_j + \mathbf{n}$ outside the Voronoi region R_j, i.e.,

$$P_{C|\mathbf{s_j}} = P\{\mathbf{r} \in R_j\}$$

Using the conditional density function $p(\mathbf{r}|\mathbf{s}_j)$, we have

$$P_{C|\mathbf{s_j}} = \int_{R_j} \frac{1}{(\pi N_o)^{N/2}} e^{-\|\mathbf{r}-\mathbf{s}_j\|^2/N_o}$$

Union Bound

In general, the Voronoi regions are very hard to determine so the integral

$$P_{C|\mathbf{s_j}} = \int_{R_j} \frac{1}{(\pi N_o)^{N/2}} e^{-\|\mathbf{r}-\mathbf{s}_j\|^2/N_o}$$

is very difficult if not impossible to compute, since we need to determine the upper and lower limits on an N-fold integral for a often complicated convex region in an N-dimensional space. In this case, upper and lower bounding techniques are useful.

Suppose we wish to compute $P_{C|\mathbf{s}_k}$.

Consider *only the pair* of signals $\mathbf{s}_k$ and $\mathbf{s}_j$. Let $\mathbf{s}_k$ be sent and let E_j denote the event that the receiver choose $\mathbf{s}_j$, hence making an error. Note that

$$P(E_j) = P(\mathbf{s}_k, \mathbf{s}_j)$$

Union Bound

The probability of symbol error for $\mathbf{s}_k$ is

$$P_{E|\mathbf{s}_k} = P\left(\bigcup_{j \neq k} E_j\right)$$

The *union bound* on $P_{E|\mathbf{s}_k}$ is

$$P\left(\bigcup_{j \neq k} E_j\right) \leq \sum_{j \neq k} P(E_j)$$

Hence,

$$P_{E|\mathbf{s}_k} \leq \sum_{j \neq k} P(\mathbf{s}_k, \mathbf{s}_j)$$

If the $\mathbf{s}_i$ are equally likely, then

$$P_M = \frac{1}{M} \sum_{k=1}^{M} P_{E|\mathbf{s}_k} \leq \frac{1}{M} \sum_{k=1}^{M} \sum_{j \neq k} P(\mathbf{s}_k, \mathbf{s}_j)$$

Union Bound

We have seen earlier that

$$P(\mathbf{s}_k, \mathbf{s}_j) = Q\left(\sqrt{\frac{d_{kj}^2}{2N_o}}\right)$$

where $d_{kj}^2 = \|\mathbf{s}_k - \mathbf{s}_j\|^2$.

Note that $Q(x)$ decreases with x. Hence, a further upper bound can be obtained by using the minimum distance $d_{\min} = \min_{j,k} d_{kj}$ and noting that

$$P(\mathbf{s}_k, \mathbf{s}_j) \;=\; Q\left(\sqrt{\frac{d_{kj}^2}{2N_o}}\right) \leq Q\left(\sqrt{\frac{d_{\min}^2}{2N_o}}\right)$$

Hence,

$$P_M \leq (M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_o}}\right)$$