EE4061
Communication Systems

Week 10

Rotations and Translations

Intersymbol Interference

Nyquist Pulse Shaping

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Signal Set Rotation

The probability of symbol error is invariant to any rotation of the signal constellation $\{\mathbf{s}_i\}_{i=1}^{M}$ about the origin of the signal space. This is a consequence of two properties.

First, the probability of symbol error depends solely on the set of Euclidean distances $\{d_{jk}\}, j \neq k$ between the signal vectors in the signal constellation.

Second, the AWGN is circularly symmetric in all directions of the signal space.

A signal constellation can be rotated about the origin of the signal space, by multiplying each N-dimensional signal vector by an $N \times N$ unitary matrix $\mathbf{Q}$. A unitary matrix has the property $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = I$, where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$, and I is the $N \times N$ identity matrix.
The rotated signal vectors are equal to

$$\hat{\mathbf{s}}_i = \mathbf{s}_i\mathbf{Q}, \quad i = 1, \ldots, M \ .$$

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Signal Set Rotation

Correspondingly, the noise vector $\mathbf{n}$ is replaced with its rotated version

$$\hat{\mathbf{n}} = \mathbf{n}\mathbf{Q} \ .$$

The rotated noise vector $\hat{\mathbf{n}}$ is a vector of complex Gaussian random variables that is completely described by its mean and covariance matrix. The mean is

$$\mathrm{E}[\hat{\mathbf{n}}] = \mathrm{E}[\mathbf{n}]\mathbf{Q} = \mathbf{0} \ .$$

The covariance matrix is

$$\begin{aligned}
\mathbf{\Lambda}_{\hat{\mathbf{n}}\hat{\mathbf{n}}} &= \mathrm{E}[\hat{\mathbf{n}}^T\hat{\mathbf{n}}] \\
&= \mathrm{E}[(\mathbf{n}\mathbf{Q})^T\mathbf{n}\mathbf{Q}] \\
&= \mathrm{E}[\mathbf{Q}^T\mathbf{n}^T\mathbf{n}\mathbf{Q}] \\
&= \mathbf{Q}^T\mathrm{E}[\mathbf{n}^T\mathbf{n}]\mathbf{Q} \\
&= \frac{N_o}{2}\mathbf{Q}^T\mathbf{Q} = \frac{N_o}{2}\mathbf{I} = \mathbf{\Lambda}_{\mathbf{nn}} \ .
\end{aligned}$$

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Signal Set Translation

Next consider a translation of the signal set such that

$$\hat{\mathbf{s}}_i = \mathbf{s}_i - \mathbf{a}, \quad i = 1, \ldots, M \ ,$$

where $\mathbf{a}$ is a constant vector. In this case, the error probability remains the same since $\hat{d}_{jk} = \tilde{d}_{jk}, j \neq k$. However, the average energy in the signal constellation is altered by the translation and becomes

$$\begin{aligned}
\hat{E}_{\text{av}} &= \sum_{i=1}^{M} \|\hat{\mathbf{s}}_i\|^2 P_i \\
&= \sum_{i=1}^{M} \|\mathbf{s}_i - \mathbf{a}\|^2 P_i \\
&= \sum_{i=1}^{M} \left(\|\mathbf{s}_i\|^2 - 2\mathbf{s}_i \cdot \mathbf{a} + \|\mathbf{a}\|^2\right) P_i \\
&= \sum_{i=1}^{M} \|\mathbf{s}_i\|^2 P_i - 2\left(\sum_{i=1}^{M} \mathbf{s}_i P_i\right) \cdot \mathbf{a} + \|\mathbf{a}\|^2 \sum_{i=1}^{M} P_i \\
&= E_{\text{av}} - 2\left(\mathrm{E}[\ \mathbf{s}\ ] \cdot \mathbf{a}\right) + \|\mathbf{a}\|^2 \qquad (1)
\end{aligned}$$

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Signal Set Translation

where E_{av} is the average energy of the original signal constellation and $\text{E}[\ \mathbf{s}\ ] = \sum_{i=0}^{M-1} \mathbf{s}_i P_i$ is its centroid (or center of mass).

Differentiating (1) with respect to the vector $\mathbf{a}$ and setting the result equal to zero will yield the translation that minimizes the average energy in the translated signal constellation. This is done by setting the N partial derivatives equal to zero, i.e.,

$$\frac{\partial \hat{E}_{\text{av}}}{\partial a_k} = -2\,[\text{E}[\ \mathbf{s}\ ]]_k\, a_k + 2a_k = 0\ , \quad k = 1, \ldots, N$$

where $[\text{E}[\ \mathbf{s}\ ]]_k$ is the k^{th} element of the vector $\text{E}[\ \mathbf{s}\ ]$. This gives

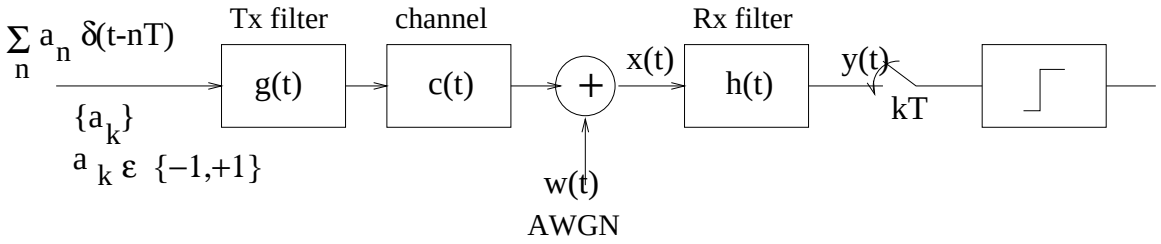
$$\mathbf{a}_{\text{opt}} = \text{E}[\ \mathbf{s}\ ]\ .$$

Note that the center of mass of the translated signal constellation is at the origin, and the minimum average energy in the translated signal constellation is

$$\hat{E}_{\text{min}} = E_{\text{av}} - \|\mathbf{a}_{\text{opt}}\|^2\ .$$

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Intersymbol Interference (ISI)

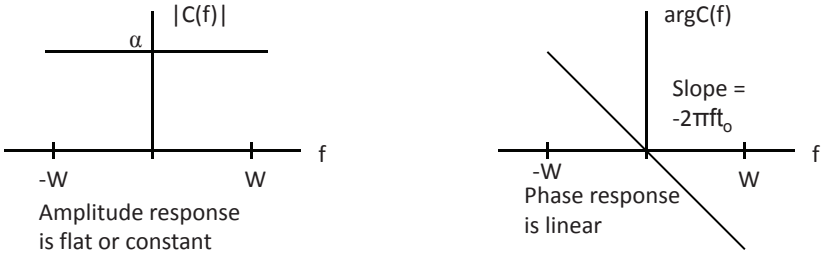


An ideal channel $c(t)$ only scales and time shifts the signal $g(t)$, but otherwise leaves it undistorted, i.e. for an ideal channel

$$\begin{aligned} c(t) &= \alpha\delta(t-t_o) \\ g(t)*c(t) &= \alpha g(t-t_o) \\ C(f) &= \alpha e^{-j2\pi f t_o} \end{aligned}$$

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ISI



Amplitude and phase response for an ideal channel.

For a more general non-ideal channel, let

$$\begin{aligned} p(t) &= g(t) * c(t) * h(t) \\ &\updownarrow \\ P(f) &= G(f)C(f)H(f) \end{aligned}$$

Then $y(t) = \sum_n a_n p(t - nT) + n(t)$, where $n(t) = w(t) * h(t)$

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ISI

$$\begin{aligned} y_i = y(iT) &= \sum_n a_n p((i-n)T) + n(iT) \\ &= \sum_n a_n p_{i-n} + n_i \end{aligned}$$

where $p_{i-n} = p((i-n)T)$, $n_i = nT$

$$y_i = a_i p_0 + \sum_{n \neq i} a_n p_{i-n} + n_i$$

$a_i p_o$: desired term,
$\sum_{n \neq i} a_n p_{i-n}$: ISI
n_i : noise

In the absence of ISI and noise, $y_i = a_i p_0$. Any analog pulse $p(t)$, such that the sampled pulse, $p_i = p(iT)$, satisfies the condition

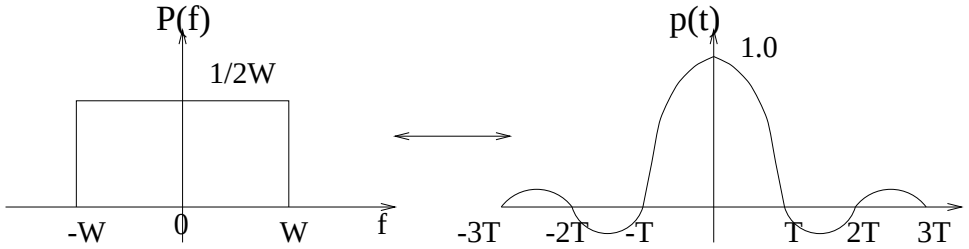
$$p_i = p(iT) = \begin{cases} p_0 & i = 0 \\ 0 & i \neq 0 \end{cases} = p_0 \delta_{i0}$$

yields zero ISI.

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Bandlimited Pulse Shaping

What overall pulse shapes $p(t)$, $p(t) = g(t) * c(t) * h(t)$, will yield zero ISI? Suppose $P(f) = \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$, where $W = 1/2T = R/2$, T is the baud duration, R is the baud rate



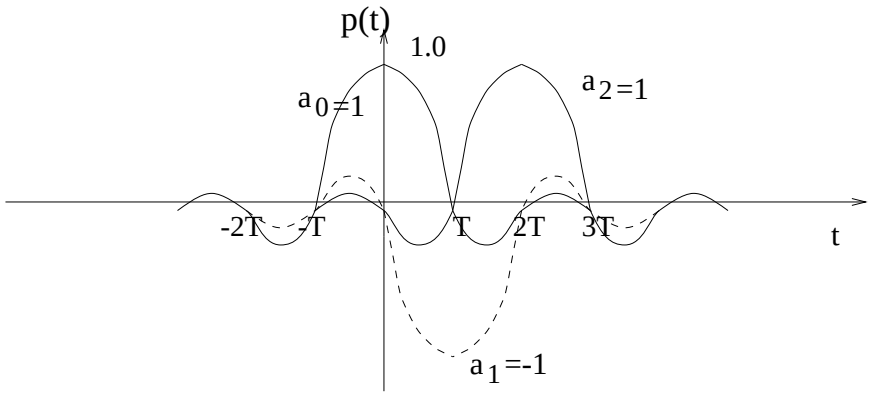
$p(t) = \text{sinc}(2Wt) = \text{sinc}(t/T)$, $W = 1/2T$.
Note that

$$p_i = p(iT) = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases} = \delta_{i0}$$

This pulse results in zero ISI. Note that $p(t)$ is noncausal in this case.

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ISI - Problems



Problems:

1. $P(f) = \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$ is an ideal low pass filter that is not realizable.
2. $p(t)$ has tails that decay slowly with time, decreasing with $1/|t|$ for large t. Therefore, $p(t)$ is very sensitive to sampler phase, i.e., a small error in the sampler timing phase can lead to significant ISI.

It is desirable to have a pulse $p(t)$ that is realizable and has tails that decay quickly in time.

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Problems with 'sinc' pulse

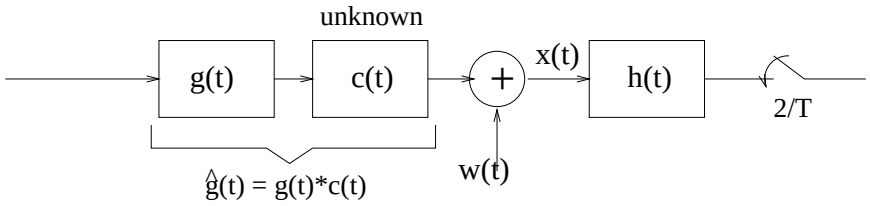
$$
\begin{aligned}
y(t) &= \sum_n a_n p(t - nT) \\
y(\Delta t) &= \sum_n a_n p(\Delta t - nT) \\
&= \sum_n a_n \frac{\sin[\pi(\Delta t - nT)/T]}{\pi(\Delta t - nT)/T} \\
&= \sum_n \frac{\sin(\pi \Delta t/T)\cos(\pi n) - \cos(\pi \Delta t/T)\sin(\pi n)}{\pi \Delta t/T - n\pi} \\
&= \sum_n a_n \frac{(-1)^n \sin(\pi \Delta t/T)}{\pi \Delta t/T - n\pi} \\
&= a_0 \mathrm{sinc}(\Delta t/T) + \frac{\sin(\pi \Delta t/T)}{\pi} \sum_{n \neq 0} \frac{a_n (-1)^n}{\Delta t/T - n}
\end{aligned}
$$

Last term is not absolutely summable, due divergence of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

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Matched Filtering and Pulse Shaping



- To maximize the signal-to-noise ratio at the output of the receiver filter $h(t)$, in theory we match the receiver filter to the received pulse $\hat{g}(t) = g(t) * c(t)$, i.e., $h(t) = \tilde{g}(T - t)$. However, if $c(t)$ is unknown, then so is $h(t)$.
- Practical Solution: Choose $h(t)$ matched to the transmitted pulse $g(t)$, i.e., choose $h(t) = g(T - t)$, over-sample by a factor of 2, and process 2 samples per baud interval.
  - This is optimal, similar to the case when $c(t)$ is known, but the proof is beyond the scope of this course.

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Matched Filtering and Pulse Shaping

- To design the transmit and receiver filters, assume an ideal channel $c(t) = \delta(t)$, so that the overall pulse (ignoring the time delay T in the matched filter) is

$$\begin{aligned} p(t) &= g(t) * h(t) \\ &= g(t) * g(-t) \end{aligned}$$

- Taking the Fourier transform of both sides

$$P(f) = G(f)G^*(f) = |G(f)|^2$$

- Hence

$$|G(f)| = \sqrt{P(f)}$$

- Many practical pulses $g(t)$ have the property that $g(t) = g(-t)$, i.e., the pulse is even in t, so that $h(t) = g(t)$. This means that the transmit and receive matched filters are identical filters.

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Conditions for ISI free transmission

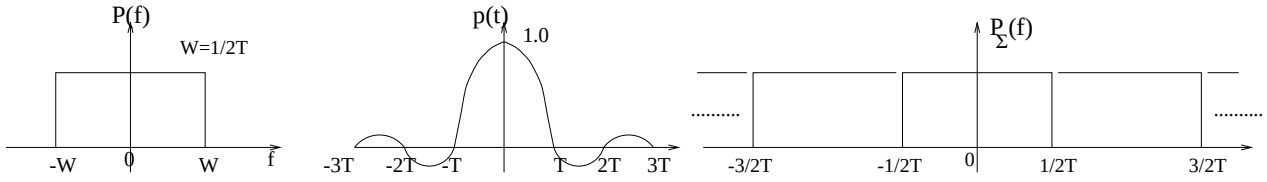
The condition for ISI-free transmission is

$$p_k = \delta_{k0} p_0 = \begin{cases} p_0 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

That is, $p(t)$ must have equally spaced zero crossings, separated by T seconds.
Theorem: The pulse $p(t)$ satisfies $p_k = \delta_{k0} p_0$ iff

$$P_{\sum}(f) \triangleq \frac{1}{T} \sum_{n=-\infty}^{\infty} P(f + n/T) = p_0$$

That is the folded spectrum $P_{\sum}(f)$ is flat.



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ISI free transmission

Proof:

$$\begin{aligned} p_k &= \int_{-\infty}^{\infty} P(f)e^{j2\pi fkT}df \\ &= \sum_{n=-\infty}^{\infty} \int_{(2n-1)/2T}^{(2n+1)/2T} P(f)e^{j2\pi fkT}df \quad f' = f - n/T \\ &= \sum_{n=-\infty}^{\infty} \int_{-1/2T}^{1/2T} P(f' + n/T)e^{j2\pi k(f'+n/T)T}df' \\ &= \int_{-1/2T}^{1/2T} e^{j2\pi f'kT} \left[\sum_{n=-\infty}^{n=\infty} P(f' + n/T) \right] df' \end{aligned} \tag{2}$$

To prove sufficiency, assume that $\sum_{n=-\infty}^{\infty} P(f' + n/T) = p_0 T$ is true. Then,

$$p_k = p_0 T \int_{-1/2T}^{1/2T} e^{j2\pi f'kT}df' = \frac{\sin \pi k}{\pi k} p_0 = \delta_{k0} p_{k0}$$

To prove necessity, from (2)

$$p_k = T \int_{-1/2T}^{1/2T} P_\Sigma(f')e^{j2\pi f'kT}df'$$

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Nyquist Pulse

Hence, p_k and $P_\Sigma(f)$ are a Fourier series pair, i.e.,

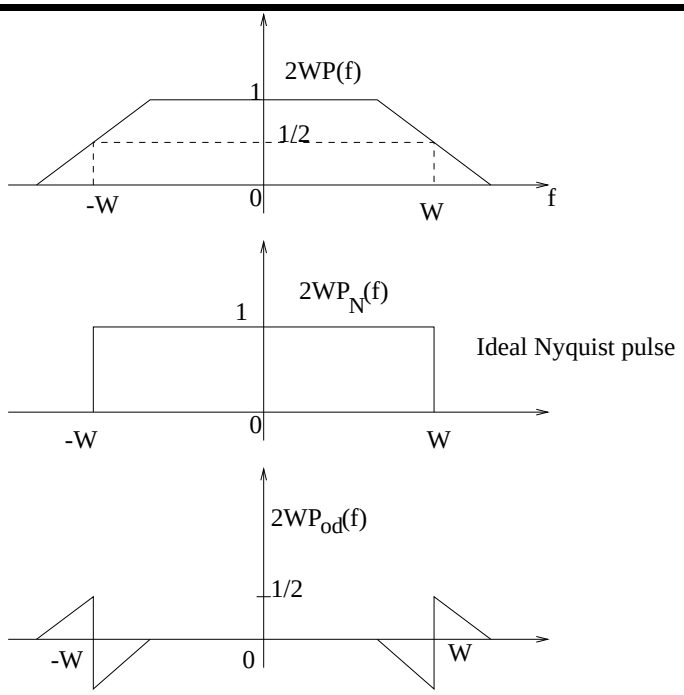
$$P_\Sigma(f) = \sum_{k=-\infty}^{\infty} p_k e^{-j2\pi fkT}$$

If $p_k = p_0\delta_{k0}$ is assumed true, then from the above equation $P_\Sigma(f) = p_0$.

- Nyquist Pulse Shaping: Any pulse $p(t)$ that yields zero-ISI is one whose folded spectrum is flat.
  - The pulse $p(t)$ can be generated by choosing $P(f)$ as shown on the following slide.

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Nyquist Pulse Shaping

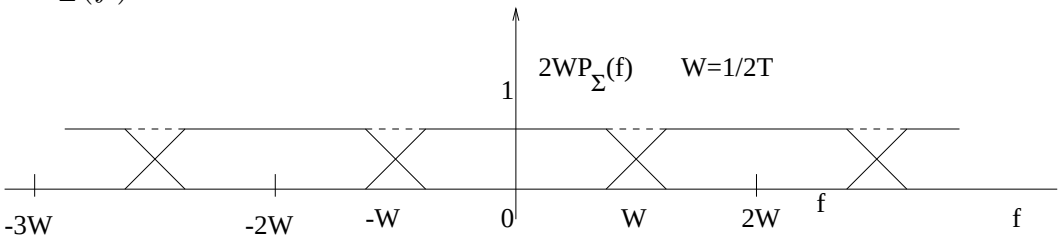


Note $P(f) = P_N(f) + P_{\rm od}(f)$.
$P_{\rm od}(f)$ can be <u>any</u> function that has skew symmetry about the Nyquist frequency $W = 1/2T$.

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Nyquist Pulse

Note that $P_\Sigma(f)$ is flat under this condition.



Example: Raised Cosine

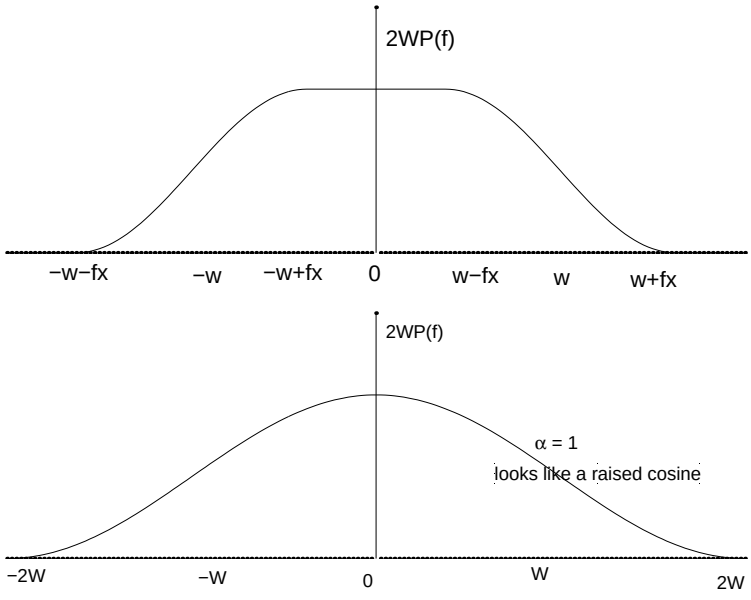
$$2WP_{\text{od}}(f) = \begin{cases} -\frac{1}{2} - \frac{1}{2}\sin\frac{\pi(|f|-W)}{2f_x} & W - f_x \le |f| \le W \\ \frac{1}{2} - \frac{1}{2}\sin\frac{\pi(|f|-W)}{2f_x} & W \le |f| \le W + f_x \end{cases}$$

f_x = bandwidth expansion, $\frac{f_x}{W} \times 100$ = excess bandwidth (%), $\alpha = \frac{f_x}{W}$ = roll off factor

$$2WP(f) = \begin{cases} 1 & 0 \le |f| \le W - f_x \\ \frac{1}{2}\left[1 - \sin\frac{\pi(|f|-W)}{2f_x}\right] & W - f_x \le |f| \le W + f_x \\ 0 & |f| \ge W + f_x \end{cases}$$

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Raised Cosine Pulse

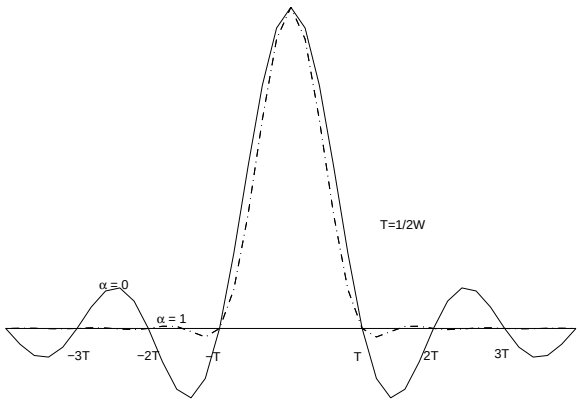


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Raised Cosine Impulse Response

Impulse response - Since $P(f)$ is even, the inverse cosine transform yields

$$\begin{aligned} p(t) &= 2\int_0^{W+f_x} P(f)\cos(2\pi ft)df \\ &= 2\frac{1}{2W}\int_0^{W-f_x}\cos(2\pi ft)df + 2\frac{1}{2W}\int_{W-f_x}^{W+f_x}\frac{1}{2}\left[1-\sin\frac{\pi|f|-W}{2f_x}\right]\cos(2\pi ft)df \\ &= \frac{\sin(2\pi Wt)}{2\pi Wt}\cdot\frac{\cos(2\pi f_x t)}{1-(4f_x t)^2} \end{aligned}$$



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Square Root Raised Cosine Pulse

- To implement a matched filter, the overall pulse $P(f)$ is split between the transmit and receive filters, i.e., $p(t) = g(t) * g(-t)$.
- However, $P(f) = G(f)G^*(f) = |G(f)|^2$, so that $|G(f)| = \sqrt{P(f)}$.
- With square-root raised cosine pulse shaping

$$\sqrt{2W}|G(f)| = \begin{cases} 1 & 0 \le |f| \le W - f_x \\ \sqrt{\frac{1}{2}\left[1 - \sin\frac{\pi(|f|-W)}{2f_x}\right]} & W - f_x \le |f| \le W + f_x \\ 0 & |f| \ge W + f_x \end{cases}$$

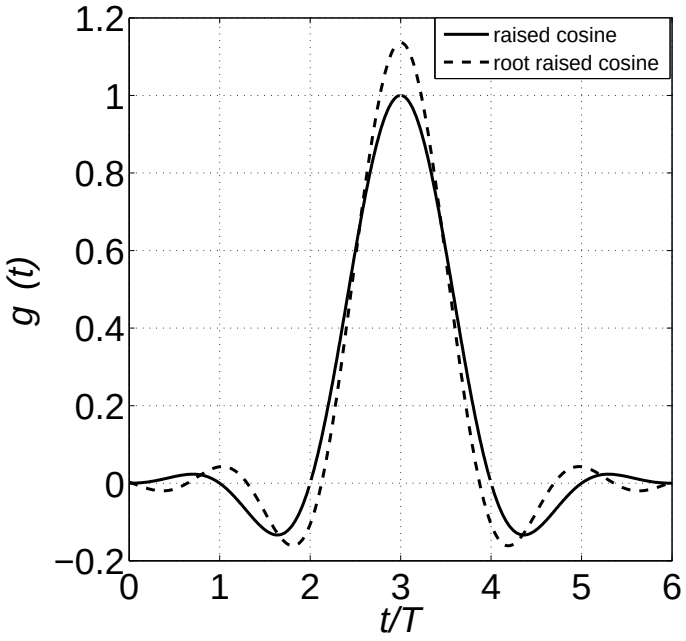
- The impulse response is

$$g(t) = \begin{cases} 1 - \alpha + 4\alpha/\pi & , \ t = 0 \\ (\alpha/\sqrt{2})\left((1 + 2/\pi)\sin(\pi/4\alpha) + (1 - 2/\pi)\cos(\pi/4\alpha)\right) & , \ t = \pm T/4\alpha \\ \frac{4\alpha(t/T)\cos((1+\alpha)\pi t/T) + \sin((1-\alpha)\pi t/T)}{\pi(t/T)(1-(4\alpha t/T)^2)} & , \ \text{elsewhere} \end{cases}$$

  where $\alpha = f_x/W$ is the rolloff factor.

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Square Root Raised Cosine Pulse



Raised cosine and root raised cosine pulses with roll-off factor $\alpha = 0.5$. The pulses are truncated to length $6T$ and time shifted by $3T$ to yield causal pulses.

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M-ary QAM

- Quadrature Amplitude Modulation (QAM), the transmitted waveform in each baud interval takes on one of the following M waveforms

$$s_m(t) = \sqrt{\frac{2E_0}{T}} g(t) \left(a_m^c \cos(2\pi f_c t) - a_m^s \sin(2\pi f_c t) \right)$$

  where

$$a_m^{\{c,s\}} \in \{\pm 1, \pm 3, \pm 5, \pm(M-1)\}$$

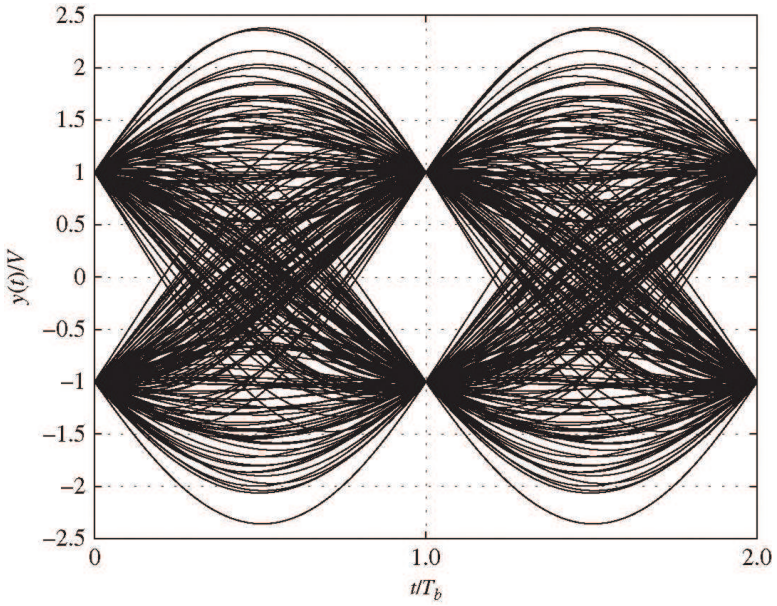
  and $2E_0$ is the energy of the signal with the lowest amplitude, i.e., when $a_m^c, a_m^s = \pm 1$.

- You have seen this before for the case $g(t) = u_T(t)$; however, practical systems will use the root-raised cosine pulse for $g(t)$. Note that we use the normalization,

$$E_g = \int_{-\infty}^{\infty} g^2(t) dt \ = \ 1 \ .$$

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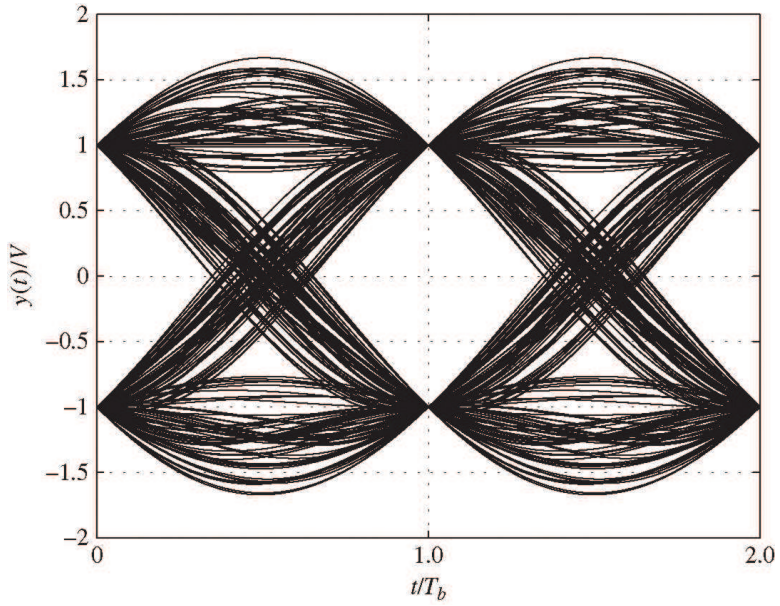
Eye Diagram with Ideal Nyquist Pulse



Eye diagram when $P(f)$ is an ideal low pass filter; $\alpha = 0$.

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Eye Diagram with Raised Cosine Pulse



Eye diagram when $P(f)$ is a raised cosine filter; $\alpha = 0.35$.

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