ECE4601
Communication Systems

Week 2

Review of Probability,
Useful Probability Distributions

Reading: Sections 3.1 to 3.10

Conditional Probability

- Consider a sample space that consists of two events A and B.
  The conditional probability $P(A|B)$ is the probability of the event A given that the event B has occurred.

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$$

- If A and B are *independent* events, then

$$P(A \bigcap B) = P(A)P(B)$$

  Hence,

$$P(A|B) = P(A)$$

  and the occurrence of event B does not change the probability of occurrence of event A.

Disjoint vs. Independent Events

- The probability of the union event "A or B" is

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$$

- If events A and B are *mutually exclusive* or *disjoint* then

$$P(A \bigcup B) = P(A) + P(B)$$

- Note that mutually exclusive and independent events are two entirely different concepts. In fact, independent events A and B with non-zero probabilities, $P(A)$ and $P(B)$, cannot be mutually exclusive because $P(A \bigcap B) = P(A)P(B) > 0$. If they were mutually exclusive, then $P(A \bigcap B) = 0$.
  - Intuitively, if the events A and B are mutually exclusive, then the occurrence of the event A precludes the occurrence of the event B. Hence, the knowledge that A has occurred definitely affects the probability that B has occurred. So A and B are not independent.

Bayes' Theorem

- Let events $A_i, i = 1, \ldots n$ be mutually exclusive such that $\bigcup_{i=1}^{n} A_i = S$, where S is the sample space. Let B be some event with non-zero probability. Then

$$\begin{aligned} P(A_i|B) &= \frac{P(A_i, B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} \end{aligned}$$

  where we use notation $P(A_i, B) = P(A_i \bigcap B)$.

- For a discrete random variable X with probability distribution function $P[X = k]$ and continuous random variable Y with probability density functions $f(y)$, the conditional probability $P[X = k|y]$ is

$$\begin{aligned} P[X = k|y] &= \frac{f(y|X = k)P[X = k]}{f(y)} \\ &= \frac{f(y|X = k)P[X = k]}{\sum_k f(y|X = k)P[X = k]} \end{aligned}$$

Bayes' Theorem - Example

- Suppose that a digital source generates 0's and 1's with unequal probabilities $Q(0) = q$ and $Q(1) = 1-q$. The bits are transmitted over a binary symmetric channel (BSC), with inputs k and outputs j, and transmission probabilities $P(j|k)$ such that $P(1|0) = P(0|1) = p$ and $P(0|0) = P(1|1) = 1-p$, where $P(j|k)$ is the probability that j is received given that k is transmitted.

- If a "1" is received what is the probability that a "0" was transmitted?

$$\begin{aligned} P(k|j) &= \frac{P(j|k)Q(k)}{P(j|0)Q(0) + P(j|1)Q(1)} \\ P(0|1) &= \frac{P(1|0)Q(0)}{P(1|0)Q(0) + P(1|1)Q(1)} \\ &= \frac{pq}{pq + (1-p)(1-q)} \end{aligned}$$

- What is the probability of bit error?

$$P_e = P(1|0)Q(0) + P(0|1)Q(1) = p$$

cdf, cdfc, and pdf

- Consider the continuous random variable X.
- The *cumulative distribution function* (cdf) of X is

$$F_X(x) = P(X \leq x) \ , \quad 0 \leq F_X(x) \leq 1$$

- The *complementary distribution function* (cdfc) of X is

$$F_X^c(x) = P(X > x) = 1 - F_X(x) \ , \quad 0 \leq F_X(x) \leq 1$$

- The *probability density function* (pdf) of X is

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

Bivariate Random Variables

- Consider two continuous random variables X and Y.
  - The *joint* (cdf) of X and Y is

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) \ , \quad 0 \leq F_{XY}(x,y) \leq 1$$

  - The *joint* (cdfc) of X and Y is

$$F^c_{XY}(x,y) = P(X > x, Y > y) = 1 - F_{XY}(x,y) \ , \quad 0 \leq F_{XY}(x,y) \leq 1$$

  - The *joint* (pdf) of X and Y is

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \qquad F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)dxdy$$

  - The *marginal* pdfs of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

  - The *conditional* pdfs of X and Y are

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Statistical Averages

- Consider the continuous random variable X.
- The mean of X is

$$\mu_X = \mathsf{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- The nth moment of X is

$$\mathsf{E}[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

- The variance of X is

$$\begin{aligned} \sigma_X^2 &= \mathsf{E}[(X-\mu_X)^2] \\ &= \mathsf{E}[X^2 - 2X\mu_X + \mu_X^2] \\ &= \mathsf{E}[X^2] - 2\mathsf{E}[X]\mu_X + \mu_X^2 \\ &= \mathsf{E}[X^2] - \mu_X^2 \end{aligned}$$

- Consider any function $g(X)$ of the random variable X. Then

$$\mathsf{E}[g^n(X)] = \int_{-\infty}^{\infty} g^n(x) f_X(x) dx$$

Joint Moments

- Consider a pair of random variables X and Y.
  - The joint moment of X and Y is

$$\mathsf{E}[X^i Y^j] = \int_{-\infty}^{\infty} x^i y^j f_{XY}(x,y)\,dx\,dy$$

  - The *covariance* of X and Y is

$$\begin{aligned} \lambda_{XY} = \mathsf{cov}[X,Y] &= \mathsf{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathsf{E}[XY] - \mathsf{E}[X]\mu_Y - \mathsf{E}[Y]\mu_X + \mu_X\mu_Y \\ &= \mathsf{E}[XY] - \mu_X\mu_Y \end{aligned}$$

  - The *correlation coefficient* of X and Y is

$$\rho = \frac{\mathsf{cov}[X,Y]}{\sigma_X \sigma_Y}$$

  - Two random variables X and Y are *uncorrelated* iff $\mathsf{cov}[X,Y] = 0$.

    Note that *independent* $\rightarrow$ *uncorrelated*.

  - Two random variables X and Y are *orthogonal* iff $\mathsf{E}[XY] = 0$.

Characteristic Functions

- Consider the continuous random variable X.
- The *characteristic* or *moment generating* function of X is

$$\Phi_X(v) = \mathsf{E}[e^{jvX}] = \int_{-\infty}^{\infty} f_X(x) e^{jvx} dx$$

  - Except for the sign of the exponent in the integrand, the characteristic function is just the Fourier transform of the pdf.
  - Taking the derivative of both sides n times and setting $v = 0$ gives

$$\left. \frac{d^n}{dv^n} \Phi_X(v) \right|_{v=0} = (j)^n \int_{-\infty}^{\infty} x^n f_X(x) dx$$

    Recognizing the integral on the R.H.S. as the nth moment, we have

$$\left. (-j)^n \frac{d^n}{dv^n} \Phi_X(v) \right|_{v=0} = \mathsf{E}[x^n]$$

  - The pdf is inverse Fourier transform (note change in sign of exponent)

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(v) e^{-jvx} dv$$

Joint Characteristic Functions

- Consider the random variables X and Y. The joint characteristic function is

$$\Phi_{XY}(v_1, v_2) = \mathsf{E}[e^{jv_1X+jv_2Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)e^{jv_1x+jv_2y}dxdy$$

- If X and Y are independent, then

$$\begin{aligned}\Phi_{XY}(v_1, v_2) &= \mathsf{E}[e^{jv_1X+jv_2Y}] \\ &= \int_{-\infty}^{\infty} f_X(x)e^{jv_1x}dx \int_{-\infty}^{\infty} f_Y(y)e^{jv_2y}dy \\ &= \Phi_X(v_1)\Phi_Y(v_2)\end{aligned}$$

- Moments can be generated according to

$$\mathsf{E}[XY] = -\frac{\partial^2\Phi_{XY}(v_1, v_2)}{\partial v_1 \partial v_2}\,|_{v_1=v_2=0}$$

  with higher order moments generated in a straight forward extension.

Functions of Single Random Variable

- Consider the continuous random variable X with pdf $f_X(x)$ and cdf $F_X(x)$. Construct the new random variable $Y = G(X)$, where $G(X)$ is a continuous-valued monotonically increasing or decreasing function.

- The pdf of Y is

$$f_Y(y) = f_X(x) \left| \frac{\mathrm{d}x}{\mathrm{d}y} \right| \quad , \quad x = G^{-1}(y)$$

Functions of Single Random Variable

- Consider the continuous uniform random variable $X \sim U[0,1]$ with pdf

$$f_X(x) = \begin{cases} 1 & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find the pdf of $Y = e^X$.

- Answer: It follows that $x = \log_e y$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = e^{-x}$$

Then

$$f_Y(y) \;=\; f_X(x) \left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = 1 \cdot \mathrm{e}^{-x} = \mathrm{e}^{-\log_e y} = \mathrm{e}^{\log_e y^{-1}} = \frac{1}{y}$$

Finally, since $0 \leq x \leq 1$,

$$f_Y(y) \;=\; \frac{1}{y}\,, \quad 1 \leq y \leq \mathrm{e}$$

Functions of Two Random Variables

- Consider the two random variables X and Y with a known joint pdf $f_{XY}(x, y)$. Let

$$U = G_1(X, Y) \quad \text{and} \quad V = G_2(X, Y)$$

be a *bivariate transformation* of the random variables X and Y into the random variables U and V. It is assumed that the transformations are one-to-one, that is each point (x, y) is mapped onto one and only one point (u, v), and the inverse transforms exist such that

$$X = H_1(U, V) \quad \text{and} \quad Y = H_2(U, V)$$

- We wish to determine the joint pdf of U and V, $f_{UV}(u, v)$, from which we can obtain the marginal density functions $f_U(u)$ and $f_V(v)$.

Functions of Two Random Variables

- The joint pdf $f_{UV}(u,v)$ has the form

$$f_{UV}(u,v) = f_{XY}(H_1(u,v), H_2(u,v))|J(u,v)|$$

where $|J(u,v)|$ is the absolute value of the *Jacobian* $J(u,v)$ of the bivariate transformation which is given by the matrix determinant

$$J(u,v) = \det \begin{bmatrix} \frac{\partial H_1(u,v)}{\partial u} & \frac{\partial H_1(u,v)}{\partial v} \\ \\ \frac{\partial H_2(u,v)}{\partial u} & \frac{\partial H_2(u,v)}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

- To determine the joint pdf $f_{UV}(u,v)$, we evaluate the joint pdf $f_{XY}(x,y)$ with x replaced by $H_1(u,v)$ and y replaced by $H_2(u,v)$, and multiply the result by the magnitude of the Jacobian $|J(u,v)|$.

Pdf of Basic Functions of Two Random Variables

- $\mathbf{U = X + Y}$: The marginal density of U is

$$f_U(u) = \int f_{XY}(v, u - v)\mathrm{d}v$$

- $\mathbf{U = X - Y}$: The marginal density of U is

$$f_U(u) = \int f_{XY}(v, v - u)\mathrm{d}v$$

- $\mathbf{U = XY}$: The marginal density of U is

$$f_U(u) = \int f_{XY}(v, u/v)|1/v|\mathrm{d}v$$

- $\mathbf{U = X/Y}$: The marginal density of U is

$$f_U(u) = \int f_{XY}(v, v/u)|v/u^2|\mathrm{d}v$$

Pdf of Discrete Random Variables

- The discrete random variable X has the n-element sample space, $S = \{x_1, x_2, \ldots, x_n\}$, where the probability that $X = x_i$ is defined to be $\mathrm{P}[X = x_i] = p_X(x_i)$, $i = 1, \ldots, n$. The probabilities $p_X(x_i)$ must satisfy the following: i) $0 \leq p_X(x_i) \leq 1$ for $1 \leq i \leq n$, and ii) $\sum_{i=1}^{n} p_X(x_i) = 1$.

- For a discrete random variable, the cdf of X is given by

$$F_X(x) = \sum_{x_i \leq x} p_X(x_i) \quad .$$

- For discrete random variables the cdf exhibits a staircase shape. As a result of these discontinuities, the cumulative distribution function is not differentiable. However, by using impulses, a discrete random variable can be treated as a continuous random variable having the pdf

$$f_X(x) = \sum_{i=1}^{n} p_X(x_i)\delta(x - x_i) \quad ,$$

  which is differentiable.

Pdf of Discrete Random Variables - Example

- Consider a fair die with faces $\{1, 2, 3, 4, 5, 6\}$. The probability distribution is

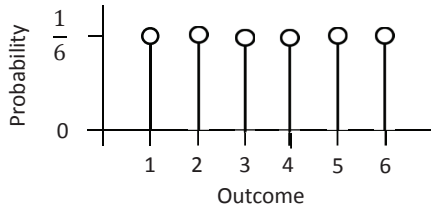
$$p_X(k) = \frac{1}{6}, \quad k = 1, 2, \ldots, 6$$

  which is depicted in the stem plot below

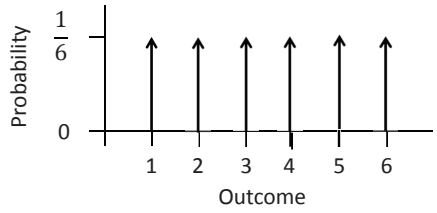
- By using impulses the probability distribution can be expressed as the pdf

$$f_X(x) = \frac{1}{6} \sum_{i=1}^{n} \delta(x - k)$$

  which is plotted below



a) Probability distribution



b) Probability density function

Binomial Distribution

- Let X be a **Bernoulli random variable** such that $X = 0$ with probability $1 - p$ and $X = 1$ with probability p. Although X is a discrete random random variable with an associated **probability distribution function**, it is possible to treat X as a continuous random variable with a **probability density function** (pdf) by using dirac delta functions.
  - The pdf of X can be written as

$$f_X(x) = (1-p)\delta(x) + p\delta(x-1)$$

- Let $Y = \sum_{i=1}^{n} X_i$, where the X_i are independent and identically distributed (iid) Bernoulli random variables. Then the random variable Y is an integer from the set $\{0, 1, \ldots, n\}$ and Y has the binomial probability distribution function

$$p_Y(k) = P[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

  Using dirac delta functions, the binomial random variable Y has the pdf

$$f_Y(y) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(y-k)$$

Poisson Distribution

- The Poisson distribution expresses the probability that a given number of events X occur in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time and/or location since the last event.

- If the average number of events in a given temporal or spatial interval is λ, then the probability of k events occurring in the same temporal or spatial interval is described by the Poisson distribution

$$p_X(k) = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} \ , \quad k = 0, 1, \ldots, \ \infty$$

- The mean and variance of the Poisson distribution are $\mu_X = \lambda$ and $\sigma_X^2 = \lambda$.

Poisson Distribution - Example

- In a circuit-switched telephone network, busy hour call attempts (BHCA) is the number of telephone calls attempted at the busiest hour of the day (peak hour). The higher the BHCA, the higher the required network capacity.

- Suppose the mean BHCA is equal to 100, as observed over many days. What is the probability that on any given day BCHA is equal to 120?

- Answer: For this problem $\lambda = 100$, so that

$$\mathrm{P}[\mathrm{BCHA} = 120] = \frac{100^{120}\mathrm{e}^{-100}}{120!} = 0.00556$$

Geometric Distribution

- The geometric distribution describes the number of failures before one success in a series of independent trials, where each trial results in either a success or failure, and the probability of success in any individual trial is equal to a constant p. The probability of success on the kth trial is described by the geometric distribution

$$p_X(k) = (1-p)^{k-1}p \ , \quad k = 1, 2, \ldots, \infty \ .$$

- The mean and variance of the geometric distribution are $\mu_X = 1/p$ and $\sigma_X^2 = (1-p)/p^2$, respectively.

Gaussian Random Variables

- A real-valued Gaussian random variable $X \sim N(\mu, \sigma^2)$ has the pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad . \quad \infty \leq x \leq \infty$$

  where $\mu = \mathsf{E}[X]$ is the mean and $\sigma^2 = \mathsf{E}[(X-\mu)^2]$ is the variance.

- The random variable $X \sim N(0,1)$ has a **standard normal density**.

- The **cumulative distribution function** (cdf) of X, $F_X(x)$, is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

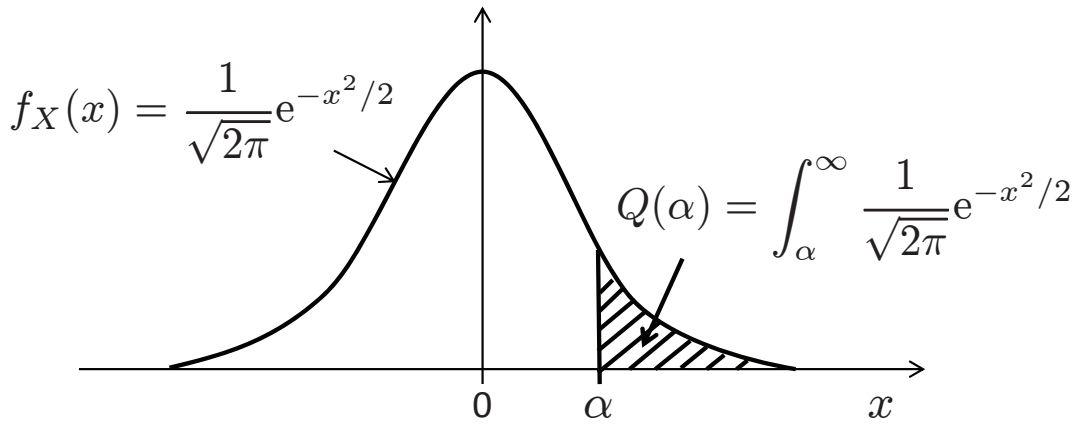
- The **complementary distribution function** (cdfc), $F_X^c(x) = 1 - F_X(x)$ of a standard normal random variable defines the Gaussian Q function

$$Q(x) \triangleq \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

  while the cdf of a standard normal random variable defines the Gaussian Φ function

$$\Phi(x) \triangleq 1 - Q(x)$$

Gaussian Q Function



Gaussian Random Variables

- If X is a non-standard normal random variable, $X \sim N(\mu, \sigma^2)$, then

$$\begin{aligned} F_X(x) &= \Phi\left(\frac{x-\mu}{\sigma}\right) \\ F_X^c(x) &= Q\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

- The **error function** $\text{erf}(x)$ and the **complementary error function** $\text{erfc}(x)$, are defined by

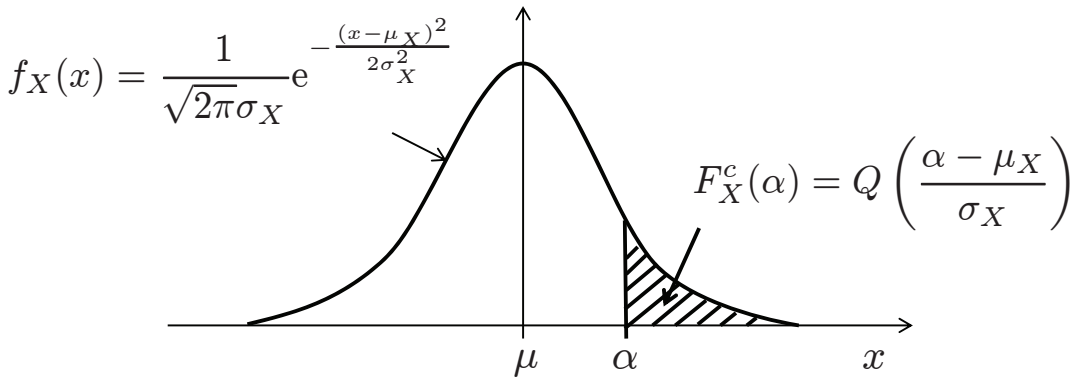
$$\text{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy \qquad \text{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

  Note that $\text{erfc}(x) \neq 1 - \text{erf}(x)$.

- The complementary error function and Gaussian Q function are related as follows

$$\begin{aligned} \text{erfc}(x) &= 2Q(\sqrt{2}x) \\ Q(x) &= \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \end{aligned}$$

Cdfc of Nonstandard Gaussian Random Variable



Multivariate Gaussian Distribution

- Let $X_i \sim N(\mu_i, \sigma_i^2), i = 1, \ldots, n$, be correlated real-valued Gaussian random variables having covariances

$$\begin{aligned} \lambda_{X_i X_j} &= \mathsf{E}\left[(X_i - \mu_i)(X_j - \mu_j)\right] \\ &= \mathsf{E}\left[X_i X_j\right] - \mu_i \mu_j, \quad 1 \le i, j \le n \end{aligned}$$

- Let

$$\begin{aligned} \mathbf{X} &= (X_1, X_2, \ldots, X_n)^T \\ \mathbf{x} &= (x_1, x_2, \ldots, x_n)^T \\ \boldsymbol{\mu}_X &= (\mu_1, \mu_2, \ldots, \mu_n)^T \\ \boldsymbol{\Lambda} &= \begin{bmatrix} \lambda_{X_1 X_1} & \cdot & \cdot & \cdot & \cdot & \lambda_{X_1 X_n} \\ \vdots & & & & & \vdots \\ \lambda_{X_n X_1} & \cdot & \cdot & \cdot & \cdot & \lambda_{X_n X_n} \end{bmatrix} \end{aligned}$$

  where $\mathbf{X}^T$ is the transpose of $\mathbf{X}$.

- The joint pdf of $\mathbf{X}$ defines the multivariate Gaussian distribution

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Lambda}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)^T \boldsymbol{\Lambda}^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right\}$$

  where $|\boldsymbol{\Lambda}|$ is the determinant of $\boldsymbol{\Lambda}$.

Bivariate Gaussian Distribution

- For the case of 2 correlated Gaussian random variables

$$\begin{aligned} \boldsymbol{\mu}_X &= (\mu_1, \mu_2)^T \\ \boldsymbol{\Lambda} &= \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \end{aligned}$$

  where $\rho = \lambda_{12}/(\sigma_1\sigma_2) = \lambda_{12}/\sigma^2$.

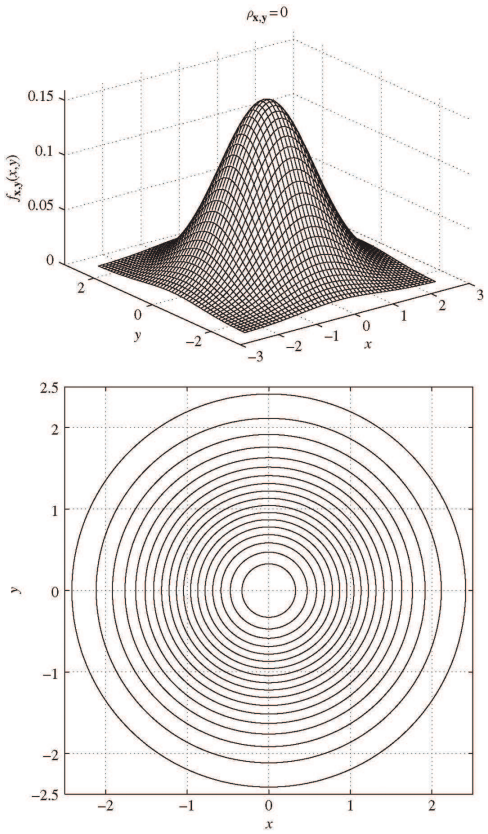
- Then $|\boldsymbol{\Lambda}| = \sigma^4(1-\rho^2)$ and

$$\boldsymbol{\Lambda}^{-1} = \frac{\sigma^2}{|\boldsymbol{\Lambda}|} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} = \frac{1}{\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

- If it happens that $\boldsymbol{\mu}_X = (0,0)$, then we have

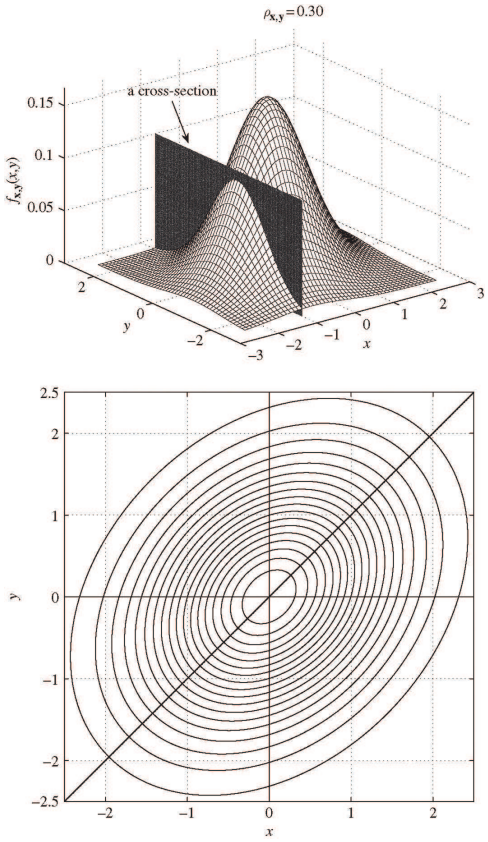
$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2\sigma^2(1-\rho^2)}(x_1,x_2)\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \\ &= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2\sigma^2(1-\rho^2)}\right] \end{aligned}$$

Bivariate Gaussian Distribution



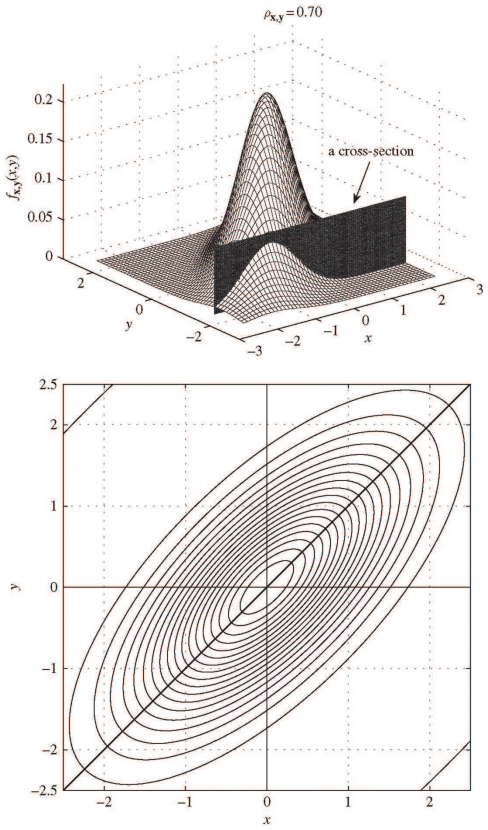
$\sigma_X = \sigma_Y = 1,\ \rho_{XY} = 0.$

Bivariate Gaussian Distribution



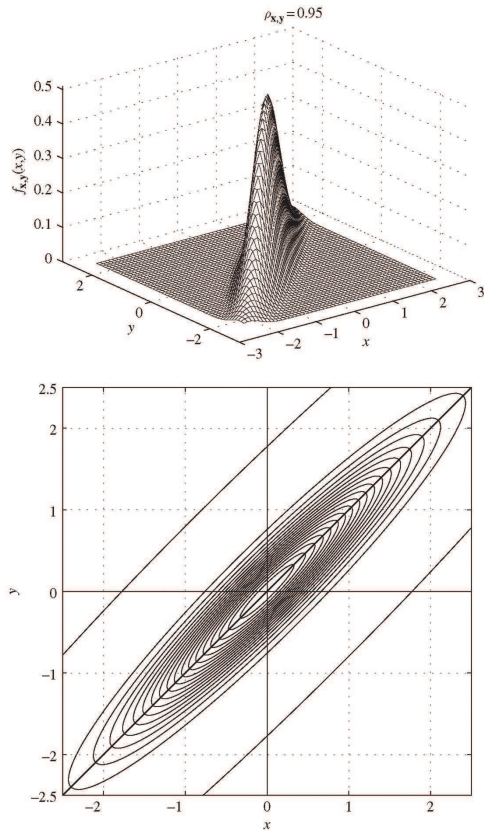
$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0.3$.

Bivariate Gaussian Distribution



$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0.7$.

Bivariate Gaussian Distribution



$\sigma_X = \sigma_Y = 1$, $\rho_{XY} = 0.95$.

Complex Gaussian Distribution

- Consider the complex Gaussian random variable $Z = X + jY$, where $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$ are independent real-valued Gaussian random variables. Such a random variable is said to be a **circularly symmetric** complex Gaussian random variable.

- The circular-symmetric complex Gaussian random variable $Z = X + jY$ has the density

$$p_Z(z) = \frac{1}{2\pi\sigma_Z^2} \exp\left\{-\frac{|z|^2}{2\sigma_Z^2}\right\} \quad ,$$

  where $\sigma_Z^2 = \frac{1}{2}\mathsf{E}[|Z|^2] = \frac{1}{2}\mathsf{E}[X^2 + Y^2] = \frac{1}{2}\mathsf{E}[X^2] + \frac{1}{2}\mathsf{E}[Y^2] = \sigma^2$.

- If $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$ are independent real-valued Gaussian random variables, then the pdf of $Z = X + jY$ is

$$p_Z(z) = \frac{1}{2\pi\sigma_Z^2} \exp\left\{-\frac{|z - \mu_Z|^2}{2\sigma_Z^2}\right\} \quad ,$$

  where $\mu_Z = \mu_X + j\mu_Y$, and $\sigma_Z^2 = \sigma_X^2 = \sigma_Y^2 = \sigma^2$.

Rayleigh Distribution

- Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \sigma^2)$ be two independent identically distributed real-valued Gaussian random variables. We wish to find the pdf of the random variable $R = \sqrt{X^2 + Y^2}$.

- For the problem at hand, we define the auxiliary variable

$$V = \mathrm{Tan}^{-1}(Y/X) \ .$$

  Then

$$\begin{aligned} X &= R\cos V \\ Y &= R\sin V \ . \end{aligned}$$

- By using a bivariate transformation of random variables

$$f_{RV}(r, v) = f_{XY}(r\cos v, r\sin v)\,|J(r, v)|$$

  where $J(r, v)$ is the Jacobian

$$J(r, v) = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \cos v & r\sin v \\ \sin v & r\cos v \end{bmatrix} = r(\cos^2 v + \sin^2 v) = r$$

Rayleigh Distribution

- Because X and Y are independent, their joint distribution has the product form

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}$$

and substituting $x = r\cos v$ and $y = r\sin v$, gives the joint pdf

$$f_{RV}(r,v) = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} \quad .$$

- To obtain the marginal pdf $f_R(r)$, the joint density is integrated over the range of the random variable V. In this case, V is an angle having range $[-\pi, \pi)$ in radian units. Hence, the marginal pdf of R is

$$f_R(r) \;=\; \int_{-\pi}^{\pi} f_{RV}(r,v)\mathrm{d}v = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} \;, \quad r \geq 0 \;\; .$$

which is a **Rayleigh distribution**.

- The cdf is

$$F_R(r) = 1 - \exp\left\{-\frac{r^2}{2\sigma^2}\right\} \;, \quad r \geq 0 \;\; .$$

Rice Distribution

- Let $X \sim \mathcal{N}(\mu_1, \sigma^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma^2)$ be independent normal random variables with non-zero means. Find the pdf of the random variable $R = \sqrt{X^2 + Y^2}$.

  As with the derivation of the Rayleigh distribution, define the auxiliary variable $V = \mathrm{Tan}^{-1}(Y/X)$. This leads to the Jacobian $J(r, v) = r$ and the bivariate transformation

$$f_{RV}(r, v) = r \cdot f_{XY}(r \cos v, r \sin v) \quad .$$

  The joint pdf of X and Y is

$$\begin{aligned} f_{XY}(x, y) &= \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{(x-\mu_1)^2 + (y-\mu_2)^2}{2\sigma^2} \right\} \\ &= \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{x^2 + y^2 + \mu_1^2 + \mu_2^2 - 2(x\mu_1 + y\mu_2)}{2\sigma^2} \right\} \quad . \end{aligned}$$

  Hence,

$$f_{RV}(r, v) = \frac{r}{2\pi\sigma^2} \exp\left\{ -\frac{r^2 + \mu_1^2 + \mu_2^2 - 2r(\mu_1 \cos v + \mu_2 \sin v)}{2\sigma^2} \right\} \quad .$$

Rice Distribution

- Now define $s \stackrel{\triangle}{=} \sqrt{\mu_1^2 + \mu_2^2}$ and $t \stackrel{\triangle}{=} \mathrm{Tan}^{-1}\mu_2/\mu_1$, $-\pi \leq t \leq \pi$, so that $\mu_1 = s\cos t$ and $\mu_2 = s\sin t$. Then

$$\begin{aligned} f_{RV}(r,v) &= \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+s^2-2rs(\cos t\cos v+\sin t\sin v)}{2\sigma^2}\right\} \\ &= \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+s^2-2rs\cos(v-t)}{2\sigma^2}\right\} \quad . \end{aligned}$$

- The marginal pdf of R is obtained by integrating over the range of V, in this case $[0, 2\pi)$,

$$f_R(r) = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2+s^2}{2\sigma^2}\right\}\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{\frac{rs}{\sigma^2}\cos(v-t)\right\}\mathrm{d}v \quad .$$

- The zero order modified Bessel function of the first kind is defined as

$$I_0(x) \stackrel{\triangle}{=} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{x\cos\theta}\mathrm{d}\theta \quad .$$

- Hence,

$$f_R(r) = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2+s^2}{2\sigma^2}\right\} I_0\left(\frac{rs}{\sigma^2}\right) \quad , \quad r \geq 0 \quad ,$$

  which is a **Rice distribution**.