EE4061
Communication Systems

Week 7

Coherent Signal Detection

Binary Modulated Signal Sets

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Optimum Coherent Detection

Suppose that we have M signals $s_1(t), s_2(t), \ldots, s_M(t)$ that are defined over the time interval $0 \leq t \leq T$, where T is the baud period or baud interval. The *baud rate* is $R = 1/T$.

Note that $T_b = T/\log_2 M$ and $R = R_b/\log_2 M$, where $R_b = 1/T_b$ is the *bit rate*.

We select one of signals, say $s_i(t)$, for transmission over an AWGN channel.

The received signal is

$$r(t) = s_i(t) + n(t)$$

where $n(t)$ is AWGN with power spectral density $N_o/2$ watts/Hz.

Problem: By observing $r(t)$ determine which one of the M signals in the set $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ was (most likely) transmitted.

Repeat this process once every T seconds for each modulated signal that is transmitted.

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Correlation Detector

Any signal set $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ can be expressed in terms of a set of orthonormal basis functions $\{f_1(t), f_2(t), \ldots, f_N(t)\}$ where N is the dimension of the signal space. Recall the Gram-Schmidt orthonormalization procedure.

However, the basis functions *do not* span the noise space, i.e., the noise waveform $n(t)$ cannot be represented exactly in terms of the N basis functions. We have

$$r(t) = \sum_{k=1}^{N} r_i f_i(t) + \hat{n}(t)$$

where

$$\begin{aligned} r_k &= \int_0^T r(t) f_k(t) dt \\ &= \int_0^T s_m(t) f_k(t) dt + \int_0^T n(t) f_k(t) dt \\ &= s_{mk} + n_k \end{aligned}$$

Hence, the projection of the received signal $r(t)$ onto the *signal space* yields the *received* vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$.

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Noise Remainder Process

Note that

$$r(t) = \sum_{k=1}^{N} r_k f_k(t) + \hat{n}(t)$$

where $\hat{n}(t)$ is the **noise remainder process**

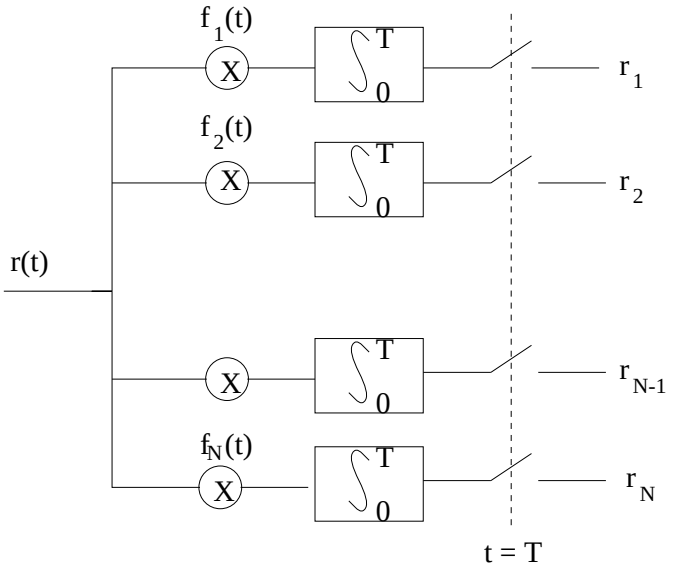
$$\hat{n}(t) = n(t) - \sum_{k=1}^{N} n_k f_k(t)$$

We will see later that the noise remainder process $\hat{n}(t)$ is irrelevant when deciding as to which $s_i(t)$ was sent.

The vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ is said to provide *sufficient statistics*, meaning that nothing else is required other than the vector $\mathbf{r}$ for the receiver to decide which $s_i(t)$ was sent.

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Correlation Detector



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Noise Statistics

The noise components n_k have mean

$$\mathrm{E}[n_k] = \int_0^T \mathrm{E}[n(t)] f_k(t) dt = 0$$

and covariance

$$\begin{aligned}
\mathrm{E}[n_j n_k] &= \int_0^T \int_0^T \mathrm{E}[n(t)n(s)] f_j(t) f_k(s) dt ds \\
&= \frac{N_o}{2} \int_0^T \int_0^T \delta(t-s) f_j(t) f_k(s) dt ds \\
&= \frac{N_o}{2} \int_0^T f_j(t) f_k(t) dt \\
&= \frac{N_o}{2} \delta_{jk}
\end{aligned}$$

Therefore, the r_k are independent Gaussian random variables with mean s_{mk} and variance $N_o/2$.

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Joint Conditional Density

Since elements of the vector $\mathbf{r}$ are independent random variables the joint conditional density function of the vector $\mathbf{r}$ has the product form

$$\begin{aligned} f(\mathbf{r}|\mathbf{s_m}) &= \prod_{k=1}^{N} f(r_k|s_{mk}) \\ &= \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_o)}} \exp\left\{-\frac{1}{N_o}(r_k - s_{mk})^2\right\} \\ &= \frac{1}{(\pi N_o)^{N/2}} \exp\left\{-\frac{1}{N_o}\sum_{k=1}^{N}(r_k - s_{mk})^2\right\} \\ &= \frac{1}{(\pi N_o)^{N/2}} \exp\left\{-\frac{1}{N_o}\|\mathbf{r} - \mathbf{s}_m\|^2\right\} \end{aligned}$$

which is a multivariate Gaussian distribution.

Note that we have used the notation

$$\|\mathbf{r} - \mathbf{s}_m\|^2 = \sum_{k=1}^{N}(r_k - s_{mk})^2$$

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Irrelevance

We have

$$\begin{aligned}
\mathrm{E}[\hat{n}(t)r_k] &= \mathrm{E}[\hat{n}(t)]s_{mk} + \mathrm{E}[\hat{n}(t)n_k] \\
&= \mathrm{E}[\hat{n}(t)n_k] \\
&= \mathrm{E}\left[\left(n(t) - \sum_{j=1}^{N} n_j f_j(t)\right) n_k\right] \\
&= \int_0^T \mathrm{E}[n(t)n(\tau)]f_k(\tau)d\tau - \sum_{j=1}^{N} \mathrm{E}[n_k n_j]f_j(t) \\
&= \frac{1}{2}N_o f_k(t) - \frac{1}{2}N_o f_k(t) = 0
\end{aligned}$$

Hence, the vector $\mathbf{r}$ is uncorrelated with $\hat{n}(t)$ and, therefore, $\hat{n}(t)$ is irrelevant since it does not contain any information about $\mathbf{r}$.

This is Wozencraft's irrelevance theorem which is certainly not irrelevant!

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Matched Filter Receiver

Suppose that we filter the received signal $r(t)$ with a bank of matched filters having the impulse responses

$$h_k(t) = f_k(T-t) \quad , \quad 0 \le t \le T$$

and sample the filter outputs at time $t = T$.

The filter outputs are

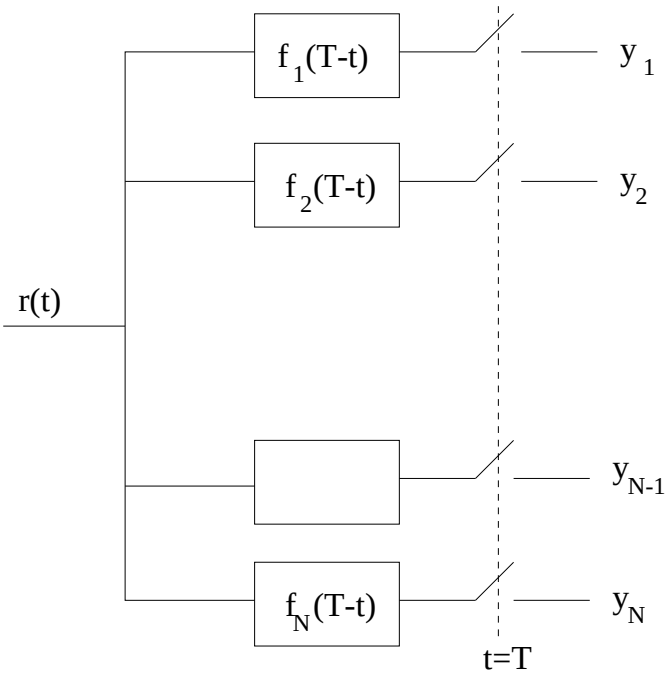
$$\begin{aligned} y_k(t) &= \int_0^t r(\tau) h_k(t-\tau) d\tau \\ &= \int_0^t r(\tau) f_k(T-t+\tau) d\tau \end{aligned}$$

$$y_k \equiv y_k(T) = \int_0^T r(\tau) f_k(\tau) d\tau$$

Note that $y_k = r_k$, i.e., the matched filter outputs are identical to the correlator outputs.

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Matched Filter Receiver



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Optimum Decision Rule

- Suppose that $s_m(t)$ is transmitted and the correlation or matched filter detector outputs the vector $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$.
- The maximum *a posteriori* (MAP) receiver observes $\mathbf{r}$ and decides in favour of the message $\mathbf{s}_k$ that maximizes the *a posteriori* probability $P(\mathbf{s}_k \text{ sent}|\mathbf{r})$.
- If $\mathbf{r}$ is received and the decision is made that $\mathbf{s}_k$ was sent, then the conditional probability of decision error is

$$P_{e|\mathbf{r}} = P(\mathbf{s}_k \text{ not sent}|\mathbf{r}) = 1 - P(\mathbf{s}_k \text{ sent}|\mathbf{r})$$

  and the unconditional probability of error is

$$P_e = \int_{-\infty}^{\infty} (1 - P(\mathbf{s}_k \text{ sent}|\mathbf{r})\, f(\mathbf{r}) d\mathbf{r}$$

- Since MAP receiver maximizes $P(\mathbf{s}_k \text{ sent}|\mathbf{r})$ for any and all $\mathbf{r}$, it minimizes the probability of error.

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Bayes' Rule Applied

- Using Bayes' rule, the *a posteriori* probability $P(\mathbf{s}_m|\mathbf{r})$ can be written in the form

$$P(\mathbf{s}_m|\mathbf{r}) = \frac{f(\mathbf{r}|\mathbf{s}_m)P_m}{f(\mathbf{r})} \quad m = 1, \ \cdots, \ M$$

- Therefore, the MAP decision rule is

$$\text{choose } \mathbf{s}_m \text{ if} \quad f(\mathbf{r}|\mathbf{s}_m)P_m \geq f(\mathbf{r}|\mathbf{s}_{\hat{m}})P_{\hat{m}} \quad \forall\ \hat{m} \neq m$$

- A receiver that chooses $\mathbf{s}_m$ to maximize $f(\mathbf{r}|\mathbf{s}_m)$ regardless of the *a priori* message probabilities P_m is called a **maximum likelihood (ML) receiver**.
- The ML decision rule is

$$\text{choose } \mathbf{s}_m \text{ if} \quad f(\mathbf{r}|\mathbf{s}_m) \geq f(\mathbf{r}|\mathbf{s}_{\hat{m}}) \quad \forall\ \hat{m} \neq m$$

- If the messages are equally likely, i.e., $\mathrm{P}_m = 1/M$, then the $\mathbf{s}_m$ that maximizes $f(\mathbf{r}|\mathbf{s}_m)$ also maximizes $P(\mathbf{s}_m|\mathbf{r})$. Under this condition, the ML receiver also minimizes the probability of error.

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Minimum Distance Decisions

With minimum distance decisions, the receiver first calculates the vector $\mathbf{r}$. The receiver then decides in favour of the signal point $\mathbf{s}_i$ that is closest in *Euclidean distance* or *squared Euclidean distance* to the received vector $\mathbf{r}$.

The minimum distance decision rule is

$$\hat{\mathbf{s}} = \underset{\mathbf{s}_i}{\arg\min} \|\mathbf{r} - \mathbf{s}_i\|^2$$

Since, the vector $\mathbf{r}$ has the joint conditional density function

$$f(\mathbf{r}|\mathbf{s}_m) = \frac{1}{(\pi N_o)^{N/2}} \exp\left\{-\frac{1}{N_o}\|\mathbf{r} - \mathbf{s}_m\|^2\right\}$$

the choice of $\mathbf{s}_i$ that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$ also maximizes the likelihood $f(\mathbf{r}|\mathbf{s}_m)$.

Hence *minimum distance* decisions are *maximum likelihood* decisions.

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Binary PSK (BPSK)

With BPSK information is transmitted in the carrier phase. Two sinusoids are used having a relative phase different of π radians. That is

$$\begin{aligned} s_1(t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t)u_T(t) \\ s_2(t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t+\pi)u_T(t) \end{aligned}$$

where $u_T(t) = u(t) - u(t-T)$ and $u(t)$ is the unit-step function. Note that

$$\begin{aligned} s_2(t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t+\pi) \\ &= \sqrt{\frac{2E}{T}}\{\cos(2\pi f_c t)\cos\pi - \sin(2\pi f_c t)\sin\pi\} \\ &= -\sqrt{\frac{2E}{T}}\cos(2\pi f_c t) = -s_1(t) \end{aligned}$$

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Binary PSK (BPSK)

Assuming that $f_cT \gg 1$, the energy in $s_1(t)$ and $s_2(t)$ is

$$\int_0^T s_i^2(t)dt = E$$

The vector representation of binary PSK signals requires only a single basis function because $s_1(t) = -s_2(t)$, i.e., the two signals are linearly dependent.

We have that

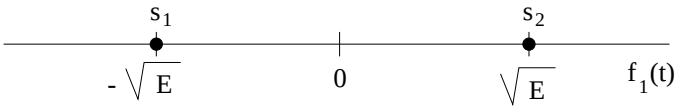
$$\begin{aligned} f_1(t) &= \frac{s_1(t)}{\sqrt{E}} \\ &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t)u_T(t) \end{aligned}$$

Then

$$\begin{aligned} s_1(t) &= \sqrt{E}f_1(t) \\ s_2(t) &= -\sqrt{E}f_1(t) \end{aligned}$$

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Binary PSK (BPSK)



$$d_{12} = 2\sqrt{E}$$

The minimum distance decision rule is

$$\text{choose} \quad \begin{cases} s_1 & \text{if } r < 0 \\ s_2 & \text{if } r > 0 \end{cases}$$

Assume s_1 is sent such that $r \sim N(s_1, N_0/2)$. The error probability is

$$\begin{aligned} P_e &= P_{e|s_1 \text{ sent}} P(s_1 \text{ sent}) + P_{e|s_2 \text{ sent}} P(s_2 \text{ sent}) \\ &= P_{e|s_1 \text{ sent}} \\ &= P(r < 0) \\ &= Q\left(\sqrt{\frac{2E}{N_o}}\right) \end{aligned}$$

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Binary FSK (BFSK)

With binary FSK signals, information is transmitted in the carrier frequency. Two sinusoids are used that have different carrier frequencies.

$$s_1(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_c t)u_T(t)$$
$$s_2(t) = \sqrt{\frac{2E}{T}}\cos(2\pi(f_c+\Delta_f)t)u_T(t)$$

The frequency difference is Δ_f.

Note that $s_1(t)$ and $s_2(t)$ both have energy E.

Depending on the choice of Δ_f, $s_1(t)$ and $s_2(t)$ may or may not be orthogonal.

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Binary FSK (BFSK)

The vector representation of binary FSK signals requires two basis functions because the two signals $s_1(t)$ and $s_2(t)$ are linearly independent.

We have that

$$\begin{aligned} f_1(t) &= \frac{s_1(t)}{\sqrt{E}} \\ &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t)u_T(t) \end{aligned}$$

Then

$$\begin{aligned} s_{21} &= \int_0^T s_2(t)f_1(t)dt \\ &= \sqrt{E}\frac{2}{T}\int_0^T \cos(2\pi(f_c+\Delta_f)t)\cos(2\pi f_c t)dt \end{aligned}$$

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Binary FSK (BFSK)

$$\begin{aligned}
s_{21} &= \frac{\sqrt{E}}{T}\int_0^T \{\cos(2\pi\Delta_f)t + \cos(2\pi(2f_c+\Delta_f)t)\}\,dt \\
&= \frac{\sqrt{E}}{T}\int_0^T \cos(2\pi\Delta_f)t\;dt \\
&= \frac{\sqrt{E}}{T}\,\frac{\sin(2\pi\Delta_f t)}{2\pi\Delta_f}\Big|_0^T \\
&= \sqrt{E}\frac{\sin(2\pi\Delta_f T)}{2\pi\Delta_f T} \\
&= \sqrt{E}\mathrm{sinc}(2\Delta_f T)
\end{aligned}$$

Hence,

$$f_2(t) = \frac{\sqrt{\frac{2E}{T}}\cos(2\pi(f_c+\Delta_f)t) - \mathrm{sinc}(2\Delta_f T)\sqrt{\frac{2E}{T}}\cos(2\pi f_c t)}{\sqrt{E(1-\mathrm{sinc}^2 2\Delta_f T)}}u_T(t)$$

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Binary Orthogonal FSK

Suppose that $\Delta_f = 1/(2T)$. Then it follows that

$$\begin{aligned} s_{21} &= \sqrt{E}\text{sinc}(2\Delta_f T) \\ &= 0 \end{aligned}$$

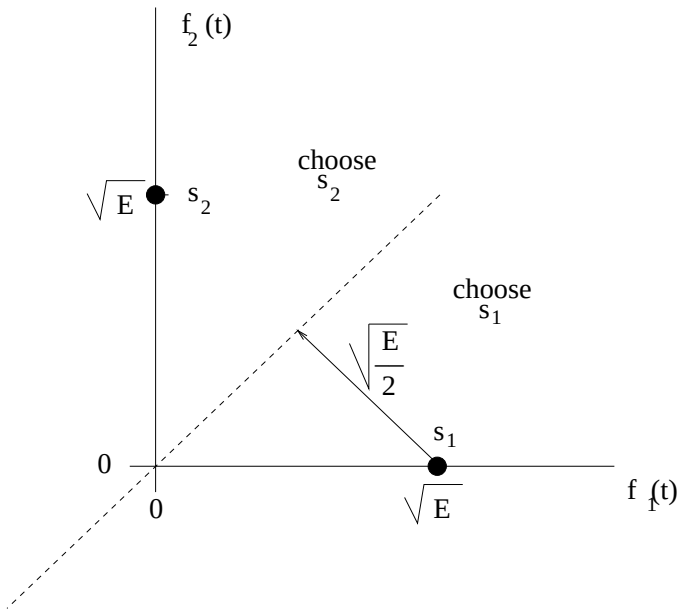
In this case, $s_1(t)$ and $s_2(t)$ are *orthogonal*.

Hence,

$$f_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi(f_c + \Delta_f)t)u_T(t)$$

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Binary Orthogonal FSK



$$d_{12}^2 = 2E$$

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Binary Orthogonal FSK

The minimum distance decision rule is

$$\text{choose} \quad \begin{cases} \mathbf{s}_1 & \text{if } \mathbf{r} \text{ is below dashed line} \\ \mathbf{s}_2 & \text{if } \mathbf{r} \text{ is above dashed line} \end{cases}$$

Using the *circular symmetric* property of the noise vector $\mathbf{n}$, the error probability is

$$\begin{aligned} P_e &= P_{e|s_1 \text{ sent}} \\ &= Q\left(\frac{\sqrt{E/2}}{\sqrt{N_o/2}}\right) \\ &= Q\left(\sqrt{\frac{E}{N_o}}\right) \end{aligned}$$

Note that coherent BFSK requires a factor of 2 (3 dB) increase in E/N_o to achieve the same error probability as BPSK.

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