Week 4

Space-time Correlation
Many mobile radio systems use antenna diversity, where spatially separated receiver antennas provide multiple faded replicas of the same information bearing signal.

The spatial decorrelation of the channel tell us the required spatial separation between antenna elements so that they will be “sufficiently” decorrelated.

Sometimes it is desirable to simultaneously characterize both the spatial and temporal correlation characteristics of the channel, e.g., when using space-time coding. This can be described by the space-time correlation function.

To obtain the spatial or space-time correlation functions, we must specify some kind of radio scattering geometry.
Single-ring scattering model for NLoS propagation on the forward link of a cellular system. The MS is surrounded by a scattering ring of radius $R$ and is at distance $D$ from the BS, where $R \ll D$. 
Model Parameters

• $O_B$: base station location
• $O_M$: mobile station location
• $D$: LoS distance from base station to mobile station
• $R$: scattering radius
• $\gamma_M$: mobile station moving direction w.r.t $x$-axis
• $v$: mobile station speed
• $\theta_M$: mobile station array orientation w.r.t. $x$-axis
• $A_M^{(i)}$: location of $i$th mobile station antenna element
• $\delta_M$: distance between mobile station antenna elements
• $S_M^{(n)}$: location of $n$th scatterer.
• $\alpha_M^{(n)}$: angle of arrival from the $n$th scatterer.
• $\epsilon_n$: distance $O_B - S_M^{(n)}$.
• $\epsilon_{ni}$: distance $S_M^{(n)} - A_M^{(i)}$. 
The channel from $O_B$ to $A_{M}^{(q)}$ has the complex envelope

$$g_q(t) = \sum_{n=1}^{N} C_n e^{j\phi_n - j2\pi(\epsilon_n + \epsilon_{nq})/\lambda_c} e^{j2\pi f_m t \cos(\alpha_M^{(n)} - \gamma_M)} , \quad q = 1, 2$$

where $\epsilon_n$ and $\epsilon_{nq}$ denote the distances $O_B - S_M^{(n)}$ and $S_M^{(n)} - A_{M}^{(q)}$, $q = 1, 2$, respectively, and $\phi_n$ is a uniform random phase on the interval $[-\pi, \pi]$.

From the Law of Cosines, the distances $\epsilon_n$ and $\epsilon_{nq}$ can be expressed as a function of the angle-of-arrival $\alpha_M^{(n)}$ as follows:

$$\epsilon_n^2 = D^2 + R^2 + 2DR \cos \alpha_M^{(n)} \quad \text{Note sign change since the angle is } \pi - \alpha_M^{(n)}$$

$$\epsilon_{nq}^2 = [(1.5 - q)\delta_M]^2 + R^2 - 2(1.5 - q)\delta_M R \cos(\alpha_M^{(n)} - \theta_M) , \quad q = 1, 2$$

Assuming that $R/D \ll 1$ (local scattering), $\delta_M \ll R$ and $\sqrt{1 \pm x} \approx 1 \pm x/2$ for small $x$, we have

$$\epsilon_n \approx D + R \cos \alpha_M^{(n)}$$

$$\epsilon_{nq} \approx R - (1.5 - q)\delta_M \cos(\alpha_M^{(n)} - \theta_M) , \quad q = 1, 2$$

Hence,

$$g_q(t) = \sum_{n=1}^{N} C_n e^{j\phi_n - j2\pi\left(D + R \cos \alpha_M^{(n)} + R - (1.5 - q)\delta_M \cos(\alpha_M^{(n)} - \theta_M)\right)/\lambda_c}$$

$$\times e^{j2\pi f_m t \cos(\alpha_M^{(n)} - \gamma_M)} , \quad q = 1, 2$$
Space-time Correlation Function

- The space-time correlation function between the two complex faded envelopes \( g_1(t) \) and \( g_2(t) \) is

\[
\phi_{g_1,g_2}(\delta_M, \tau) = \frac{1}{2} \mathbb{E}[g_1^*(t)g_2(t + \tau)]
\]

- The space-time correlation function between \( g_1(t) \) and \( g_2(t) \) can be written as

\[
\phi_{g_1,g_2}(\delta_M, \tau) = \Omega_p \sum_{n=1}^{N} \mathbb{E}\left[ e^{j2\pi(\delta_M/\lambda_c) \cos(\alpha_M^{(n)} - \theta_M)} e^{-j2\pi f_m \tau \cos(\alpha_M^{(n)} - \gamma_M)} \right].
\]

- Since the number of scatters is infinite, the discrete angles-of-arrival \( \alpha_M^{(n)} \) can be replaced with a continuous random variable \( \alpha_M \) with probability density function \( p(\alpha_M) \).

- Hence, the space-time correlation function becomes

\[
\phi_{g_1,g_2}(\delta_M, \tau) = \frac{\Omega_p}{2} \int_{0}^{2\pi} e^{jb \cos(\alpha_M - \theta_M)} e^{-ja \cos(\alpha_M - \gamma_M)} p(\alpha_M) d\alpha_M.
\]

where \( a = 2\pi f_m \tau \) and \( b = 2\pi \delta_M/\lambda_c \).
2-D Isotropic Scattering

• For the case of 2-D isotropic scattering with an isotropic receive antennas, \( p(\alpha_M) = 1/(2\pi), -\pi \leq \alpha_M \leq \pi \), and the space-time correlation function becomes

\[
\phi_{g_1,g_2}(\delta_M, \tau) = \frac{\Omega_p}{2} J_0 \left( \sqrt{a^2 + b^2 - 2ab \cos(\theta_M - \gamma_M)} \right).
\]

• The spatial and temporal correlation functions can be obtained by setting \( \tau = 0 \) and \( \delta_M = 0 \), respectively. This gives

\[
\phi_{g_1,g_2}(\delta_M) = \phi_{g_1,g_2}(\delta_M, 0) = \frac{\Omega_p}{2} J_0(2\pi \delta_M/\lambda_c)
\]

\[
\phi_{g_1,g_2}(0, \tau) = \phi_{g_1,g_2}(0, \tau) = \frac{\Omega_p}{2} J_0(2\pi f_m \tau)
\]

• Finally, we note that

\[
f_m \tau = \frac{v \cdot \tau}{\lambda_c} = \frac{\delta_M}{\lambda_c}
\]

For this scattering environment, the normalized time \( f_m \tau \) is equivalent to the normalized distance \( \delta_M/\lambda_c \).

• The antenna branches are uncorrelated if they are separated by \( \delta_M \approx 0.5\lambda_c \).
Temporal and spatial correlation functions at the MS with 2-D isotropic scattering and an isotropic receiver antenna. Note that $f_m \tau = \delta_M / \lambda_c$. 
Spatial Correlation at the Base Station

Single-ring scattering model for NLoS propagation on the reverse link of a cellular system. The MS is surrounded by a scattering ring of radius $R$ and is at distance $D$ from the BS, where $R \ll D$. 
Model Parameters

- $O_B$: base station location
- $O_M$: mobile station location
- $D$: LoS distance from base station to mobile station
- $R$: scattering radius
- $\gamma_M$: mobile station moving direction w.r.t. $x$-axis
- $v$: mobile station speed
- $\theta_B$: base station array orientation w.r.t. $x$-axis
- $A_B^{(i)}$: location of $i$th base station antenna element
- $\delta_B$: distance between mobile station antenna elements
- $S_M^{(m)}$: location of $m$th scatterer.
- $\alpha_M^{(m)}$: angle of departure to the $n$th scatterer.
- $\epsilon_m$: distance $S_M^{(m)} - O_B$.
- $\epsilon_{mi}$: distance $S_M^{(m)} - A_B^{(i)}$. 
The channel from $O_M$ to $A^{(q)}_B$ has the complex envelope

$$g_q(t) = \sum_{m=1}^{N} C_m e^{j\phi_m - j2\pi(R+\epsilon_{mq})/\lambda_c} e^{j2\pi f_m t \cos(\alpha^{(m)}_M - \gamma_M)}, \quad q = 1, 2$$

(1)

where $\epsilon_{mq}$ denote the distance $S^{(m)}_M - A^{(q)}_B, \quad q = 1, 2$, and $\phi_m$ is a uniform random phase on $(-\pi, \pi]$. To proceed further, we need to express $\epsilon_{mq}$ as a function of $\alpha^{(m)}_M$.

Applying the Law of Cosines to the triangle $\triangle S^{(m)}_M O_B A^{(q)}_B$, the distance $\epsilon_{mq}$ can be expressed as a function of the angle $\theta^{(m)}_B - \theta_B$ as follows:

$$\epsilon^2_{mq} = [(1.5 - q)\delta_B]^2 + \epsilon^2_m - 2(1.5 - q)\delta_B\epsilon_m \cos(\theta^{(m)}_B - \theta_B), \quad q = 1, 2$$

(2)

where $\epsilon_m$ is the distance $S^{(m)}_M - O_B$.

By applying the Law of Sines to the triangle $\triangle O_M S^{(m)}_M O_B$ we obtain following identity

$$\frac{\epsilon_m}{\sin \alpha^{(m)}_M} = \frac{R}{\sin \left(\pi - \theta^{(m)}_B\right)} = \frac{D}{\sin \left(\pi - \alpha^{(m)}_M - \left(\pi - \theta^{(m)}_B\right)\right)}.$$
• Since the angle $\pi - \theta_B^{(m)}$ is small, we can apply the small angle approximations $\sin x \approx x$ and $\cos x \approx 1$ for small $x$, to the second equality in the above identity. This gives

$$\frac{R}{(\pi - \theta_B^{(m)})} \approx \frac{D}{\sin(\pi - \alpha_M^{(m)})}$$

or

$$(\pi - \theta_B^{(m)}) \approx (R/D) \sin(\pi - \alpha_M^{(m)}).$$

• It follows that the cosine term in (2) becomes

$$\cos(\theta_B^{(m)} - \theta_B) = \cos(\pi - \theta_B - (\pi - \theta_B^{(m)}))$$

$$= \cos(\pi - \theta_B) \cos(\pi - \theta_B^{(m)}) + \sin(\pi - \theta_B) \sin(\pi - \theta_B^{(m)})$$

$$\approx \cos(\pi - \theta_B) + \sin(\pi - \theta_B)(R/D) \sin(\pi - \alpha_M^{(m)})$$

$$= -\cos(\theta_B) + (R/D) \sin(\theta_B) \sin(\alpha_M^{(m)})$$

(3)

• Using the approximation in (3) in (2), along with $\delta_B/\epsilon_m \ll 1$, gives

$$\epsilon_{mq}^2 \approx \epsilon_m^2 \left[ 1 - 2(1.5 - q)\frac{\delta_B}{\epsilon_m} \left( \frac{R}{D} \sin(\theta_B) \sin(\alpha_M^{(m)}) - \cos(\theta_B) \right) \right].$$
• Applying the approximation $\sqrt{1 \pm x} \approx 1 \pm x/2$ for small $x$, we have

$$\epsilon_{mq} \approx \epsilon_m - (1.5 - q)\delta_B \left[ (R/D) \sin(\theta_B) \sin(\alpha_M^{(m)}) - \cos(\theta_B) \right] . \tag{4}$$

• Applying the Law of Cosines to the triangle $\triangle O_M S_M O_B$ we have

$$\epsilon_m^2 = D^2 + R^2 - 2DR \cos(\alpha_M^{(m)})$$

$$\approx D^2 \left[ 1 - 2(R/D) \cos(\alpha_M^{(m)}) \right] ,$$

and again using the approximation $\sqrt{1 \pm x} \approx 1 \pm x/2$ for small $x$, we have

$$\epsilon_m \approx D - R \cos(\alpha_M^{(m)}) \tag{5}$$

• Finally, using (5) in (4) gives

$$\epsilon_{mq} \approx D - R \cos(\alpha_M^{(m)}) - (1.5 - q)\delta_B \left[ (R/D) \sin(\theta_B) \sin(\alpha_M^{(m)}) - \cos(\theta_B) \right] . \tag{6}$$

• Substituting (6) into (1) gives the result

$$g_q(t) = \sum_{m=1}^{N} C_m e^{j\phi_m + j2\pi f_m t} \cos(\alpha_M^{(m)} - \gamma_M)$$

$$\times e^{-j2\pi \left(R + D - R \cos(\alpha_M^{(m)}) - (1.5 - q)\delta_B \left[ (R/D) \sin(\theta_B) \sin(\alpha_M^{(m)}) - \cos(\theta_B) \right] \right)/\lambda_c} ,$$

which no longer depends on the $\epsilon_{mq}$ and is a function of the angle of departure $\alpha_M^{(m)}$. 
The space-time correlation function between the two complex faded envelopes $g_1(t)$ and $g_2(t)$ at the BS is once again given by

$$
\phi_{g_1,g_2}(\delta_B, \tau) = \frac{1}{2} \mathbb{E}[g_1^*(t)g_2(t + \tau)]
$$

Using (7), the space-time correlation function between $g_1(t)$ and $g_2(t)$ can be written as

$$
\phi_{g_1,g_2}(\delta_B, \tau) = \frac{\Omega_p}{2N} \sum_{m=1}^{N} \mathbb{E}[e^{j2\pi(\delta_B/\lambda_c)}(R/D)\sin(\theta_B)\sin(\alpha_M^{(m)}) - \cos(\theta_B)]
\times e^{-j2\pi f_m \tau \cos(\alpha_M^{(m)} - \gamma_M)}. 
$$

Since the number of scatters around the MS is infinite, the discrete angles-of-departure $\alpha_M^{(m)}$ can be replaced with a continuous random variable $\alpha_M$ with probability density function $p(\alpha_M)$.

Hence, the space-time correlation function becomes.

$$
\phi_{g_1,g_2}(\delta_B, \tau) = \frac{\Omega_p}{2} \int_{-\pi}^{\pi} e^{-ja\cos(\alpha_M - \gamma_M)} e^{jb[(R/D)\sin(\theta_B)\sin(\alpha_M) - \cos(\theta_B)]} p(\alpha_M) d\alpha_M,
$$

where $a = 2\pi f_m \tau$ and $b = 2\pi \delta_B / \lambda_c$. 

For the case of 2-D isotropic scattering with an isotropic MS transmit antenna, $p(\alpha_M) = 1/(2\pi), -\pi \leq \alpha_M \leq \pi$, and the space-time correlation function becomes

$$\phi_{g1,g2}(\delta_B, \tau) = \frac{\Omega_p}{2} e^{-jb\cos(\theta_B)}$$
$$\times J_0\left(\sqrt{a^2 + b^2(R/D)^2 \sin^2(\theta_B) - 2ab(R/D) \sin(\theta_B) \sin(\gamma_M)}\right).$$

The spatial and temporal correlation functions can be obtained by setting $\tau = 0$ and $\delta_B = 0$, respectively.

The temporal correlation function $\phi_{gg}(\tau) = \phi_{g1,g2}(0, \tau) = \frac{\Omega_p}{2} J_0(2\pi f_m \tau)$ which matches our result for the received signal at a mobile station.

The spatial correlation function is

$$\phi_{g1,g2}(\delta_B) = \phi_{g1,g2}(\delta_B, 0) = \frac{\Omega_p}{2} e^{-jb\cos(\theta_B)} J_0\left(b(R/D) \sin(\theta_B)\right)$$
$$= \frac{\Omega_p}{2} e^{-jb\cos(\theta_B)} J_0\left((2\pi \delta_B/\lambda_c)(R/D) \sin(\theta_B)\right).$$

Observe that a much greater spatial separation is required to achieve a given degree of envelope decorrelation at the BS as compared to the MS. This can be readily seen by the term $R/D \ll 1$ in the argument of the Bessel function.
Envelope crosscorrelation magnitude at the base station for different base station antenna orientation angles, $\theta_B$; $D = 3000$ m, $R = 60$ m. Broadside base station antennas have the lowest crosscorrelation.
Envelope crosscorrelation magnitude at the base station for $\theta_B = \pi/3$ and various scattering radii, $R$; $D = 3000$ m. Smaller scattering radii will result in larger crosscorrelations.